ROUGH SETS DETERMINED BY TOLERANCES

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ABSTRACT. We show that for any tolerance R on U, the ordered sets of lower and upper rough approximations determined by R form ortholattices. These ortholattices are completely distributive, thus forming atomistic Boolean lattices, if and only if R is induced by an irredundant covering of U, and in such a case, the atoms of these Boolean lattices are described. We prove that the ordered set RS of rough sets determined by a tolerance R on U is a complete lattice if and only if it is a complete subdirect product of the complete lattices of lower and upper rough approximations. We show that R is a tolerance induced by an irredundant covering of U if and only if RS is an algebraic completely distributive lattice, and in such a situation a quasi-Nelson algebra can be defined on RS. We present necessary and sufficient conditions which guarantee that for a tolerance R on U, the ordered set RS_X is a lattice for all $X \subseteq U$, where R_X denotes the restriction of R to the set X and RS_X is the corresponding set of rough sets. We introduce the disjoint representation and the formal concept representation of rough sets, and show that they are Dedekind–MacNeille completions of RS.

1. INTRODUCTION

Rough sets were introduced in [15] by Z. Pawlak. The key idea is that our knowledge about the properties of the objects of a given universe of discourse U may be inadequate or incomplete in the sense that the objects of the universe U can be observed only within the accuracy of indiscernibility relations. According to Pawlak's original definition, an indiscernibility relation E on U is an equivalence relation interpreted so that two elements of U are E-related if they cannot be distinguished by their properties known by us. Thus, indiscernibility relations allow us to partition a set of objects into classes of indistinguishable objects. For any subset $X \subseteq U$, the *lower approximation* $X^{\blacktriangledown}$ of X consists of elements such that their E-class is included in X, and the *upper approximation* $X^{\blacktriangle}$ of X is the set of the elements whose E-class intersects with X. This means that $X^{\blacktriangledown}$ can be viewed as the set of elements certainly belonging to X, because all elements E-related to them are also in X. Similarly, $X^{\blacktriangle}$ may be interpreted as the set of elements that possibly are in X, because in X there is at least one element indiscernible to them. The *rough set* of X is the pair $(X^{\blacktriangledown}, X^{\blacktriangle})$ and the set of all rough sets is

$$RS = \{(X^{\blacktriangledown}, X^{\blacktriangle}) \mid X \subseteq U\}.$$

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The set RS may be canonically ordered by the coordinatewise order:

$$(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle}) \iff X^{\blacktriangledown} \subseteq Y^{\blacktriangledown} \text{ and } X^{\blacktriangle} \subseteq Y^{\blacktriangle}.$$

In [16] it was proved that RS is a lattice which forms also a Stone algebra. Later this result was improved in [2] by showing that RS is in fact a regular double Stone algebra. Therefore, RS determines also a three-valued Łukasiewicz algebra and a semi-simple Nelson algebra, because it is well known that these three types of algebras can be transformed to each other [14].

In the literature can be found numerous generalizations of rough sets such that equivalences are replaced by relations of different types. For instance, it is known that in the case of quasiorders (reflexive and transitive binary relations), a Nelson algebra such that the underlying rough set lattice is an algebraic lattice can be defined on RS [11, 12]. If rough sets are determined by relations that are symmetric and transitive, then the structure of RS is analogous to the case of equivalences [8].

In this paper, we assume that indiscernibility relations are tolerances (reflexive and symmetric binary relations). The term *tolerance relation* was introduced in the context of visual perception theory by E. C. Zeeman [21], motivated by the fact that indistinguishability of "points" in the visual world is limited by the discreteness of retinal receptors. One can argue that tolerances suit better for representing indistinguishability than equivalences, because transitivity is the least obvious property of indiscernibility. Namely, we may have a finite sequence of objects $x_1, x_2, \ldots, x_n$ such that each two consecutive objects x_i and x_{i+1} are indiscernible, but there is a notable difference between x_1 and x_n. It is known [6, 7] that in the case of tolerances, RS is not necessarily a lattice if the cardinality of U is greater than four. Our main goals in this work are to find conditions under which RS forms a lattice, and, in case RS is a lattice, to study its properties.

The paper is organized as follows: In Section 2, we present the definition of rough approximation operators and present their essential properties. In addition, we give preliminaries of Galois connections, ortholattices, and formal concepts. Section 3 is devoted to the rough set operators defined by tolerance relations. Starting from the well-known fact that for any tolerance on U, the pair $({}^{\blacktriangle}, {}^{\blacktriangledown})$ is a Galois connection on the power set lattice of U and characterize rough set approximation pairs as certain kind of Galois connections (F, G) on a power set. We show that $\wp(U)^{\blacktriangledown} = \{X^{\blacktriangledown} \mid X \subseteq U\}$ and $\wp(U)^{\blacktriangle} = \{X^{\blacktriangle} \mid X \subseteq U\}$ form ortholattices and prove that these ortholattices are completely distributive if and only if R is induced by an irredundant covering of U. Note that distributive ortholattices are Boolean lattices, and a Boolean lattice is atomistic if and only if it is completely distributive. This means that $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are atomistic Boolean lattices exactly when R is induced by an irredundant covering of U, and we describe the atoms of these lattices. In Section 4, we study the ordered set of rough sets RS and show that it can be up to isomorphism identified with a set of pairs $\{(\mathcal{I}(X), \mathcal{C}(X)) \mid X \subseteq U\}$, where $\mathcal{I}$ and $\mathcal{C}$ are interior and closure operators on the set U satisfying certain conditions. We prove that RS is a complete lattice if and only if it is a complete subdirect product of $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$. We also show that RS is an algebraic completely distributive lattice if and only if R is induced by an irredundant covering of U, and in such a case, on RS a quasi-Nelson algebra can be defined. The section ends with necessary and sufficient conditions which guarantee that for a tolerance R on U, the ordered set RS_X is a lattice for all $X \subseteq U$, where R_X denotes the restriction of R to the set X and RS_X is the set of all rough sets determined by R_X.

Finally, Section 5 is devoted to the disjoint representation and the formal concept representation of rough sets. In particular, we prove that these representations are Dedekind–MacNeille completions of RS.

2. Preliminaries: Rough approximation operators, Galois connections, and formal concepts

First we recall from [9] some notation and basic properties of rough approximation operators defined by arbitrary binary relations. Let R be a binary relation on the set U. For any $X \subseteq U$, we denote

$$R(X) = \{y \in U \mid x\,R\,y \text{ for some } x \in X\}.$$

For the singleton sets, $R(\{x\})$ is written simply as $R(x)$, that is, $R(x) = \{y \in U \mid x\,R\,y\}$. It is clear that $R(X) = \bigcup_{x \in X} R(x)$ for all $X \subseteq U$. The *lower approximation* of a set $X \subseteq U$ is

$$X^{\blacktriangledown} = \{x \mid R(x) \subseteq X\}$$

and X's *upper approximation* is

$$X^{\blacktriangle} = \{x \mid R(x) \cap X \neq \emptyset\}.$$

Let $\wp(U)$ denote the *power set* of U. It is a complete Boolean lattice with respect to the set-inclusion order. The map $^{\blacktriangle}$ is a complete join-homomorphism on $\wp(U)$, that is, it preserves all unions:

$$\Big(\bigcup_{X \in \mathcal{H}} X\Big)^{\blacktriangle} = \bigcup_{X \in \mathcal{H}} X^{\blacktriangle}.$$

Analogously, $^{\blacktriangledown}$ is a complete meet-homomorphism on $\wp(U)$ preserving all intersections:

$$\Big(\bigcap_{X \in \mathcal{H}} X\Big)^{\blacktriangledown} = \bigcap_{X \in \mathcal{H}} X^{\blacktriangledown}.$$

Hence, the approximation operators are order-preserving, that is, $X \subseteq Y$ implies $X^{\blacktriangledown} \subseteq Y^{\blacktriangledown}$ and $X^{\blacktriangle} \subseteq Y^{\blacktriangle}$. In addition, approximation operators are dual, meaning that for all $X \subseteq U$, $X^{c\blacktriangle} = X^{\blacktriangledown c}$ and $X^{c\blacktriangledown} = X^{\blacktriangle c}$, where X^c denotes the set-theoretical *complement* $U \setminus X$ of X. By the above, the set

$$\wp(U)^{\blacktriangledown} = \{X^{\blacktriangledown} \mid X \subseteq U\}$$

is a *closure system*, that is, it is closed under arbitrary intersections. Similarly,

$$\wp(U)^{\blacktriangle} = \{X^{\blacktriangle} \mid X \subseteq U\}$$

forms an interior system, that is, it is closed under any union. The complete lattices $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are with respect to the set-inclusion relation dually order-isomorphic by the map $X^{\blacktriangledown} \mapsto X^{\blacktriangledown c} = X^{c\blacktriangle}$.

For two ordered sets P and Q, a pair (f, g) of maps $f\colon P \to Q$ and $g\colon Q \to P$ is called a *Galois connection* between P and Q if for all $p \in P$ and $q \in Q$,

$$f(p) \le q \iff p \le g(q).$$

In the next lemma are listed some of the known properties of Galois connections.

Lemma 2.1. *Let (f, g) be a Galois connection between two ordered sets P and Q.*

(a) *The composition $f \circ g \circ f$ equals f and the composition $g \circ f \circ g$ equals g.*

(b) *The composition $g \circ f$ is a lattice-theoretical closure operator on P and the set of $g \circ f$-closed elements is $g(Q)$, that is, $(g \circ f)(P) = g(Q)$.*

(c) *The composition $f \circ g$ is a lattice-theoretical interior operator on Q and the set of $f \circ g$-closed elements is $f(P)$, that is, that is, $(f \circ g)(Q) = f(P)$.*
(d) *The image sets $f(P)$ and $g(Q)$ are order-isomorphic.*
(e) *The map f is a complete join-homomorphism and g is a complete meet-homomorphism.*
(f) *The maps f and g uniquely determine each other by the equations*

$$f(p) = \bigwedge\{q \in Q \mid p \leq g(q)\} \quad \textit{and} \quad g(q) = \bigvee\{p \in P \mid f(p) \leq q\}.$$

In addition, if the maps $f\colon P \to Q$ and $g\colon Q \to P$ between two complete lattices P and Q form a Galois connection (f, g), then $f(P)$ is a complete lattice such that for all $S \subseteq f(P)$,

$$\bigvee S = \bigvee\nolimits_Q S \quad \text{and} \quad \bigwedge S = f\big(g\big(\bigwedge\nolimits_Q S\big)\big) = f\big(\bigwedge\nolimits_P g(S)\big),$$

and $g(Q)$ is a complete lattice such that for all $S \subseteq g(Q)$,

$$\bigvee S = g\big(f\big(\bigvee\nolimits_P S\big)\big) = g\big(\bigvee\nolimits_Q f(S)\big) \quad \text{and} \quad \bigwedge S = \bigwedge\nolimits_P S.$$

An *orthocomplementation* on a bounded lattice is a function that maps each element x to an *orthocomplement* $x^\perp$ in such a way that the following axioms hold:
(O1) $x \leq y$ implies $y^\perp \leq x^\perp$;
(O2) $x^{\perp\perp} = x$;
(O3) $x \vee x^\perp = 1$ and $x \wedge x^\perp = 0$.
An *ortholattice* is a bounded lattice equipped with an orthocomplementation. Ortholattices are self-dual by the map $^\perp$. Note that if an ortholattice is distributive, then it is a Boolean lattice such that the complement of the element x is $x^\perp$.

Let (f, g) be a Galois connection on a Boolean lattice $(B, \vee, \wedge, ^c, 0, 1)$. We may define the maps $^\perp\colon f(B) \to f(B)$ and $^\top\colon g(B) \to g(B)$ by setting

$$f(x)^\perp = f(f(x)^c) \quad \text{and} \quad g(x)^\top = g(g(x)^c).$$

The maps $^\perp$ and $^\top$ satisfy (O1), and if f and g are dual, that is, $f(x)^c = g(x^c)$ for all $x \in B$, then $^\perp$ and $^\top$ satisfy (O2). Additionally, if f is extensive, that is, $x \leq f(x)$ for all $x \in B$, then $^\perp$ and $^\top$ satisfy (O3). These observations are summarized in the following well-known lemma.

Lemma 2.2. *If (f, g) is a Galois connection on a Boolean lattice B such that f and g are dual, and f is extensive, then $f(B)$ and $g(B)$ are ortholattices.*

We end this section by presenting some terminology concerning formal concepts from [4]. A *formal context* $\mathbb{K} = (G, M, I)$ consists of two sets G and M and a relation I from G to M. The elements of G are called the *objects* and the elements of M are called *attributes* of the context. We write $g\,I\,m$ or $(g, m) \in I$ to mean that the object g has the attribute m. For $A \subseteq G$ and $B \subseteq M$, we define

$$A' = \{m \in M \mid g\,I\,m \text{ for all } g \in A\} \quad \text{and} \quad B' = \{g \in G \mid g\,I\,m \text{ for all } m \in B\}.$$

A *formal concept* of the context (G, M, I) is a pair (A, B) with $A \subseteq G$, $B \subseteq M$, $A' = B$, and $B' = A$. We call A the *extent* and B the *intent* of the concept (A, B). It is easy to see that $(A, B) \in \wp(G) \times \wp(M)$ is a concept if and only if $(A, B) = (A'', A') = (B', B'')$. The set of all concepts of the context $\mathbb{K} = (G, M, I)$ is denoted by $\mathfrak{B}(\mathbb{K})$. The set $\mathfrak{B}(\mathbb{K})$ is ordered by

$$(A_1, B_1) \leq (A_2, B_2) \iff A_1 \subseteq A_2 \iff B_1 \supseteq B_2. \tag{2.1}$$

With respect to this order, $\mathfrak{B}(\mathbb{K})$ forms a complete lattice, called the *concept lattice* of the context $\mathbb{K}$, in which

$$\bigvee_{j\in J}(A_j,B_j)=\Big(\big(\bigcup_{j\in J}A_j\big)'',\bigcap_{j\in J}B_j\Big) \quad\text{and}\quad \bigwedge_{j\in J}(A_j,B_j)=\Big(\bigcap_{j\in J}A_j,\big(\bigcup_{j\in J}B_j\big)''\Big).$$

3. Approximation operations defined by tolerances

In this section, we are recalling from [6, 9] the characteristic properties of rough sets approximation operators defined by *tolerance relations*, which are reflexive and symmetric binary relations. Also some new results are presented. It is known that the relation R is reflexive if and only if $X^{\blacktriangledown}\subseteq X\subseteq X^{\blacktriangle}$ for all $X\subseteq U$. Similarly, R is symmetric if and only if the pair $({}^{\blacktriangle},{}^{\blacktriangledown})$ is a Galois connection on $\wp(U)$. In the rest of this section, we assume that R is a tolerance on U. Note that for all $X\subseteq U$, we have $X^{\blacktriangle}=R(X)=\bigcup_{x\in X}R(x)$.

Proposition 3.1. *Let (F,G) be a Galois connection on the complete lattice $\wp(U)$. Then, there exists a tolerance R on U such that F equals ${}^{\blacktriangle}$ and G equals ${}^{\blacktriangledown}$ if and only if the following conditions hold for all $x,y\in U$:*

(i) $x\in F(\{x\})$;
(ii) $x\in F(\{y\})$ *implies* $y\in F(\{x\})$.

Proof. ($\Rightarrow$) Suppose that F equals ${}^{\blacktriangle}$ and G equals ${}^{\blacktriangledown}$ for some tolerance R. Then, condition (i) means that $x\in R(x)$ for all $x\in X$ and (ii) is equivalent to that $x\in R(y)$ implies $y\in R(x)$ for any $x,y\in U$. These conditions are obviously satisfied, because R is a tolerance.

($\Leftarrow$) Let us define a binary relation R by setting $x\,R\,y$ if and only if $x\in F(\{y\})$. Because F satisfies (i) and (ii), and $R(x)=F(\{x\})$ for all $x\in U$, the relation R is a tolerance. In addition,

$$X^{\blacktriangle}=\bigcup_{x\in X}R(x)=\bigcup_{x\in X}F(\{x\})=F\big(\bigcup_{x\in X}\{x\}\big)=F(X),$$

because F is a complete join-homomorphism. Since the pairs of maps forming Galois connections are unique by Lemma 2.1(f), we have that G must equal ${}^{\blacktriangledown}$. □

Because $({}^{\blacktriangle},{}^{\blacktriangledown})$ is a Galois connection on $\wp(U)$, the approximation operators have all the properties listed in Lemma 2.1. In particular, the map $X\mapsto X^{\blacktriangle\blacktriangledown}$ is the closure operator corresponding to the closure system $\wp(U)^{\blacktriangledown}$, which forms a complete lattice with respect to the order $\subseteq$ such that

$$\bigvee_{X\in\mathcal{H}}X^{\blacktriangledown}=\big(\bigcup_{X\in\mathcal{H}}X^{\blacktriangledown}\big)^{\blacktriangle\blacktriangledown}\quad\text{and}\quad\bigwedge_{X\in\mathcal{H}}X^{\blacktriangledown}=\bigcap_{X\in\mathcal{H}}X^{\blacktriangledown}\tag{3.1}$$

for all $\mathcal{H}\subseteq\wp(U)$. In addition, $\wp(U)^{\blacktriangledown}=\{X^{\blacktriangle\blacktriangledown}\mid X\subseteq U\}$.

Analogously, the map $X\mapsto X^{\blacktriangledown\blacktriangle}$ is the interior operator that corresponds the interior system $\wp(U)^{\blacktriangle}$, which is a complete lattice such that

$$\bigvee_{X\in\mathcal{H}}X^{\blacktriangle}=\bigcup_{X\in\mathcal{H}}X^{\blacktriangle}\quad\text{and}\quad\bigwedge_{X\in\mathcal{H}}X^{\blacktriangle}=\big(\bigcap_{X\in\mathcal{H}}X^{\blacktriangle}\big)^{\blacktriangledown\blacktriangle}\tag{3.2}$$

for all $\mathcal{H}\subseteq\wp(U)$. The closure system $\wp(U)^{\blacktriangle}$ can be also written in the form $\{X^{\blacktriangledown\blacktriangle}\mid X\subseteq U\}$.

By Lemma 2.2, $\wp(U)^\blacktriangle$ and $\wp(U)^\blacktriangledown$ are ortholattices. In $\wp(U)^\blacktriangle$, the orthocomplementation is ${}^\perp \colon X^\blacktriangle \mapsto X^{\blacktriangle c \blacktriangle}$, and the map ${}^\top \colon X^\blacktriangledown \mapsto X^{\blacktriangledown c \blacktriangledown}$ is the orthocomplementation operation of $\wp(U)^\blacktriangledown$. Hence, $\wp(U)^\blacktriangle$ and $\wp(U)^\blacktriangledown$ are self-dual, and

$$(\wp(U)^\blacktriangle, \subseteq) \cong (\wp(U)^\blacktriangle, \supseteq) \cong (\wp(U)^\blacktriangledown, \subseteq) \cong (\wp(U)^\blacktriangledown, \supseteq).$$

Next we study the relationship between the lattices of approximations and concept lattices. For a tolerance R on a set U, we consider the context $\mathbb{K} = (U, U, R^c)$, whose concept lattice is $\mathfrak{B}(\mathbb{K}) = \{(X'', X') \mid X \subseteq U\}$. For $X \subseteq U$,

$$X' = \{x \in U \mid y \, R^c \, x \text{ for all } y \in X\} = \{x \in U \mid (x, y) \notin R \text{ for all } y \in X\} = X^{\blacktriangle c}.$$

Thus, $X^\blacktriangle = X'^{\,c}$ and $X^\blacktriangledown = X^{c \blacktriangle c} = X^{c\,\prime}$. In addition, $X'' = X^{\blacktriangle\blacktriangledown}$ and hence

$$\mathfrak{B}(\mathbb{K}) = \{(X^{\blacktriangle\blacktriangledown}, X^{c\blacktriangledown}) \mid X \subseteq U\}.$$

If $(A, B) \in \mathfrak{B}(\mathbb{K})$, then $A, B \in \wp(U)^\blacktriangledown$ such that $A = B'$ and $B = A'$. For any $A \in \wp(U)^\blacktriangledown$, $A' = A^{\blacktriangle c} = A^{c\blacktriangledown} = A^\top$, where ${}^\top$ is the orthocomplement defined in $\wp(U)^\blacktriangledown$. Thus, $(A, B) = (A, A^\top) = (B^\top, B)$. On the other hand, if $A \in \wp(U)^\blacktriangledown$, then $A = A^{\blacktriangle\blacktriangledown}$ and $(A, A^\top)$ belongs to $\in \mathfrak{B}(\mathbb{K})$. Hence,

$$\mathfrak{B}(\mathbb{K}) = \{(A, A^\top) \mid A \in \wp(U)^\blacktriangledown\}.$$

Let $\wp(U)^{\blacktriangledown \mathrm{op}}$ denote the dual of the lattice $\wp(U)^\blacktriangledown$, that is, $(\wp(U)^\blacktriangledown, \supseteq)$.

Proposition 3.2. *Let R be a tolerance on a set U and $\mathbb{K} = (U, U, R^c)$.*

(a) *The complete lattices $\wp(U)^\blacktriangle$, $\wp(U)^\blacktriangledown$, and $\mathfrak{B}(\mathbb{K})$ are isomorphic.*

(b) *The concept lattice $\mathfrak{B}(\mathbb{K})$ is a complete sublattice of $\wp(U)^\blacktriangledown \times \wp(U)^{\blacktriangledown \mathrm{op}}$.*

Proof. (a) It is obvious that the map $A \mapsto (A, A^\top)$ is an isomorphism between $\wp(U)^\blacktriangledown$ and $\mathfrak{B}(\mathbb{K})$, and we have already noted that $\wp(U)^\blacktriangledown$ and $\wp(U)^\blacktriangle$ are isomorphic.

(b) Clearly, $\mathfrak{B}(\mathbb{K}) \subseteq \wp(U)^\blacktriangledown \times \wp(U)^\blacktriangledown$. For all $\{A_j\}_{j \in J} \subseteq \wp(U)^\blacktriangledown$, the join in $\wp(U)^\blacktriangledown$ is $\bigvee_{j \in J} A_j = (\bigcup_{j \in J} A_j)^{\blacktriangle\blacktriangledown} = (\bigcup_{j \in J} A_j)''$, and the meet in $\wp(U)^\blacktriangledown$ is $\bigwedge_{j \in J} A_j = \bigcap_{j \in J} A_j$. Thus, the join in $\wp(U)^{\blacktriangledown \mathrm{op}}$ is $\bigvee_{j \in J} A_j = \bigcap_{j \in J} A_j$. Therefore, for any $\{(A_j, B_j)\}_{j \in J} \subseteq \mathfrak{B}(\mathbb{K})$, its join $\bigvee_{j \in J} (A_j, B_j)$ coincides in $\mathfrak{B}(\mathbb{K})$ and $\wp(U)^\blacktriangledown \times \wp(U)^{\blacktriangledown \mathrm{op}}$. An analogous observation can be done with respect to meets. Thus, $\mathfrak{B}(\mathbb{K})$ is a complete sublattice of $\wp(U)^\blacktriangledown \times \wp(U)^{\blacktriangledown \mathrm{op}}$. □

Note that Proposition 3.2 implies that for a tolerance R and for the context $\mathbb{K} = (U, U, R^c)$, the concept lattice $\mathfrak{B}(\mathbb{K})$ is an ortholattice and the orthocomplement is obtained by swapping the sets (cf. [4, p. 54]), in other words, for $(A, A^\top) \in \mathfrak{B}(\mathbb{K})$, its orthocomplement is $(A^\top, A)$.

The lattice $\wp(U)^\blacktriangledown$ is not necessarily even modular; for instance, in Example 4.3 (p. 11), we define a tolerance R on the set $U = \{a, b, c, d, e\}$ such that $\wp(U)^\blacktriangledown = \{\emptyset, \{a\}, \{c\}, \{e\}, \{a, b\}, \{a, e\}, \{d, e\}, \{a, b, c\}, \{c, d, e\}, U\}$. Now, the set $\{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, b, c\}\}$ forms a sublattice of $\wp(U)^\blacktriangledown$ isomorphic to $\mathbf{N_5}$.

A complete lattice L is *completely distributive* if for any doubly indexed subset $\{x_{i,\,j}\}_{i \in I,\, j \in J}$ of L, we have

$$\bigwedge_{i \in I} \Big(\bigvee_{j \in J} x_{i,\,j} \Big) = \bigvee_{f \colon I \to J} \Big(\bigwedge_{i \in I} x_{i,\,f(i)} \Big),$$

that is, any meet of joins may be converted into the join of all possible elements obtained by taking the meet over $i \in I$ of elements $x_{i,\,k}$, where k depends on i.

In [4], Theorem 40 presents the following condition equivalent to the assertion that the concept lattice $\mathfrak{B}(\mathbb{K})$ is completely distributive:

(†) For every non-incident object-attribute pair $(g,m) \notin I$, there exists an object $h \in G$ and an attribute $n \in M$ with $(g,n) \notin I, (h,m) \notin I$, and $h \in k''$ for all $k \in G \setminus \{n\}'$.

We know by Proposition 3.2 that for any tolerance R on U, $\wp(U)^{\blacktriangle}$ and $\wp(U)^{\blacktriangledown}$ are isomorphic to the concept lattice of the context $\mathbb{K} = (U, U, R^c)$. If $\mathbb{K}$ is identified with (G, M, I), then for all $x, y \in U$, $(x,y) \notin I$ means that $x \,R\, y$ and $y \,R\, x$. Since $x'' = \{x\}^{\blacktriangle\blacktriangledown} = R(x)^{\blacktriangledown}$, $y \in x''$ means that $R(y) \subseteq R(x)$. In addition, $U \setminus \{x\}' = \{x\}'^{c} = \{x\}^{\blacktriangle} = R(x)$. Hence (†) is equivalent to the following condition:

(‡) For any $a \,R\, b$, there exist $c, d \in U$ with $a \,R\, c$ and $b \,R\, d$ such that for all $k \in R(c)$, we have $R(d) \subseteq R(k)$.

In what follows, we are going to present some conditions equivalent to (‡). Let R be a tolerance on U. A set $X \subseteq U$ is a *preblock* of R if $a \,R\, b$ holds for all $a, b \in X$, that is, $X^2 \subseteq R$, where X^2 means the Cartesian product $X \times X$. A *block* of R is a maximal preblock B. It is well known that $B = \bigcap_{x \in B} R(x)$, and that any preblock is contained in some block of R (see e.g. [18]). Hence, for any $x, y \in U$, $x \,R\, y$ if and only if there exists a block B such that $x, y \in B$.

Denoting the set of blocks of R by $\mathcal{B}(R)$, we obtain $R = \bigcup \{B^2 \mid B \in \mathcal{B}(R)\}$. A collection $\mathcal{H} \subseteq \wp(U)$ of nonempty subsets of U is called a *covering* of U if $\bigcup \mathcal{H} = U$. A covering $\mathcal{H}$ is *irredundant* if $\mathcal{H} \setminus \{X\}$ is not a covering of U for any $X \in \mathcal{H}$. Clearly, the blocks of any tolerance on form a covering, which is not in general irredundant. Conversely, for any covering $\mathcal{H}$ of U, the relation $R_{\mathcal{H}} = \bigcup \{X^2 \mid X \in \mathcal{H}\}$ is tolerance on U, called the *tolerance induced by* $\mathcal{H}$.

Lemma 3.3. *Let R be a tolerance on U. The following assertions are equivalent:*

(a) *R satisfies (‡).*

(b) *For any $a \,R\, b$, there exists $d \in U$ with $R(d) \subseteq R(a) \cap R(b)$ such that for all $x \,R\, d$, we have $R(d) \subseteq R(x)$.*

(c) *For any $a \,R\, b$, there exists a block $B \in \mathcal{B}(R)$ and an element $d \in B$ such that $a, b \in R(d) = B$.*

(d) *R is a tolerance induced by an irredundant covering of U.*

Proof. (a)⇒(b): Let R be a tolerance satisfying (‡). Then, $a \,R\, b$ implies that there are $c \in R(a)$ and $d \in R(b)$ such that $R(d) \subseteq R(c)$ and $R(d) \subseteq R(a)$. Additionally, $b \in R(d)$ implies $b \in R(c)$, which gives $R(d) \subseteq R(b)$. Thus, $R(d) \subseteq R(a) \cap R(b)$. Finally, if $x \,R\, d$, then $x \in R(d) \subseteq R(c)$, and hence $R(d) \subseteq R(x)$ by (‡).

(b)⇒(c): Suppose R satisfies (b). If $a \,R\, b$, then there is an element $d \in U$ with $R(d) \subseteq R(a) \cap R(b)$. Thus, also $a \,R\, d$ and $b \,R\, d$ hold, and we have $\{a, b, d\}^2 \subseteq R$. Hence, there is a block $B \in \mathcal{B}(R)$ with $a, b, d \in B$. Note that since R satisfies (b), we have $x \,R\, d$ and $R(d) \subseteq R(x)$ for all $x \in B$. Thus, we get

$$B \subseteq R(d) \subseteq \bigcap_{x \in B} R(x) = B,$$

and hence $a, b \in R(d) = B$.

(c)⇒(d): Suppose that (c) holds and let us define a family $\mathcal{K}$ of blocks by $\mathcal{K} = \{B \in \mathcal{B}(R) \mid B = R(d) \text{ for some } d \in U\}$. Now, for all $x \in U$, $x \,R\, x$ implies that there is a block $B \in \mathcal{K}$ containing x, and so $\mathcal{K}$ is a covering of U. In view of (c), for any $a \,R\, b$, there is a block $B \in \mathcal{K}$ with $a, b \in B$. Hence, $R \subseteq \bigcup_{B \in \mathcal{K}} B^2 \subseteq R$, giving $R = \bigcup_{B \in \mathcal{K}} B^2$. Thus, R is induced by the covering $\mathcal{K}$. Finally, we show that $\mathcal{K}$ is irredundant by proving that $\bigcup (\mathcal{K} \setminus \{B\}) \neq U$ for any $B \in \mathcal{K}$. Indeed, if $B \in \mathcal{K}$,

then there exists $d \in U$ such that $R(d) = B$. Suppose that $d \in X$ for some block $X \in \mathcal{K} \setminus \{B\}$. Then, $d\,R\,x$ for all $x \in X$, whence we get $X \subseteq R(d) = B$. Since X and B are blocks, we obtain $X = B$, a contradiction. Thus, $d \notin \bigcup(\mathcal{K} \setminus \{B\})$.

(d)⇒(b): Assume that $R = \bigcup\{X^2 \mid X \in \mathcal{H}\}$, where $\mathcal{H}$ is an irredundant covering of U and suppose that $a\,R\,b$. Then, there exists $X \in \mathcal{H}$ such that $a, b \in X$, and clearly, $x\,R\,y$ for all $x, y \in X$. Hence, $X \subseteq R(x)$ for all $x \in X$. Since $\bigcup(\mathcal{H} \setminus \{X\}) \neq U$, there is $d \in X$ such that $d \notin Y$ for all $Y \in \mathcal{H} \setminus \{X\}$. Observe that $d\,R\,y$ for some $y \in U \setminus X$ would imply that $\{d, y\}$ is contained in some block $Y \in \mathcal{H}$ different from X. Since this is impossible, we get $R(d) \subseteq X \subseteq R(x)$ for all $x \in X$. In particular, we obtain $R(d) \subseteq R(a) \cap R(b)$, and also $R(d) \subseteq R(x)$ for each $x \in U$ with $x\,R\,d$, because $x\,R\,d$ implies $x \in R(d) \subseteq X$.

(b)⇒(a): If (b) holds, then (‡) is satisfied with $c = d$. □

Our next proposition is now clear by the above-mentioned observations.

Proposition 3.4. *Let R be a tolerance on a set U. The complete lattices $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are completely distributive if and only if R is induced by an irredundant covering of U.*

Remark 3.5. If $a\,R\,b$ holds for any $a, b \in R(x)$, then $R(x)$ is a block. Namely, in this case $R(x) = \bigcap_{a \in R(x)} R(a)$, which means that $R(x)$ is block of R.

Let L be a lattice with a least element 0. The lattice L is *atomistic*, if any element of L is the join of atoms below it. It is well known (see e.g. [5]) that a complete Boolean lattice is atomistic if and only if it is completely distributive.

Proposition 3.6. *Let R be a tolerance induced by an irredundant covering of U.*

(a) *$\wp(U)^{\blacktriangledown}$ is an atomistic Boolean lattice such that $\{R(x)^{\blacktriangledown} \mid R(x)$ is a block$\}$ is the set of atoms.*
(b) *$\wp(U)^{\blacktriangle}$ is an atomistic Boolean lattice such that $\{R(x) \mid R(x)$ is a block$\}$ is the set of atoms.*

Proof. By Proposition 3.4, $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are completely distributive. Because they are ortholattices also, they are Boolean lattices, and since they are completely distributive, they must be atomistic.

Atoms of $\wp(U)^{\blacktriangle}$ need to be of the form $R(x)$, because the map $^{\blacktriangle}$ is order-preserving and $R(x) = \{x\}^{\blacktriangle}$. Since $^{\blacktriangledown}$ is an isomorphism from $\wp(U)^{\blacktriangle}$ to $\wp(U)^{\blacktriangledown}$, the atoms of $\wp(U)^{\blacktriangledown}$ are of the form $R(x)^{\blacktriangledown}$.

Suppose that $R(x)$ is a block and $R(y) \subseteq R(x)$. Because $y \in R(x)$ and $R(x)$ is a block, we must have $R(y) = R(x)$, and hence $R(x)$ is an atom of $\wp(U)^{\blacktriangle}$. Analogously, $R(x)^{\blacktriangledown}$ is an atom of $\wp(U)^{\blacktriangledown}$, whenever $R(x)$ is a block.

On the other hand, suppose $R(x)$ is an atom of $\wp(U)^{\blacktriangle}$; then $R(x)^{\blacktriangledown}$ is an atom of $\wp(U)^{\blacktriangledown}$. Suppose that $a, b \in R(x)$. Since $a\,R\,x$, by Lemma 3.3, there exists $d \in U$ with $R(d) \subseteq R(a) \cap R(x)$, and so $d \in R(a)^{\blacktriangledown} \cap R(x)^{\blacktriangledown}$. Hence, $\emptyset \subset R(a)^{\blacktriangledown} \cap R(x)^{\blacktriangledown} \subseteq R(x)^{\blacktriangledown}$, and because $R(x)^{\blacktriangledown}$ is an atom, we have $R(a)^{\blacktriangledown} \cap R(x)^{\blacktriangledown} = R(x)^{\blacktriangledown}$, that is, $R(x)^{\blacktriangledown} \subseteq R(a)^{\blacktriangledown}$. Now $x \in R(x)^{\blacktriangledown} \subseteq R(a)^{\blacktriangledown}$ implies $b \in R(x) \subseteq R(a)$. Thus, $a\,R\,b$, and so, by Remark 3.5, $R(x)$ is a block. This completes the proof of both cases (a) and (b). □

Example 3.7. Let $A = \{1, 2, ..., n\}$ be a finite set. We define a tolerance R on $U = \wp(A) \setminus \{\emptyset\}$ by setting for any nonempty subsets $B, C \subseteq A$:

$$(B, C) \in R \iff B \cap C \neq \emptyset.$$

The structure (U,R) is called an $n-1$-*dimensional simplex* (see [18]). Let $i \in A$ and define the set $K_i = \{B \in U \mid i \in B\}$. Clearly, $R(\{i\}) = K_i$, and it is easy to see that K_i is also a tolerance block. Now let $\mathcal{H} = \{K_1, K_2, ..., K_n\}$. Then, $(B,C) \in R$ means that B and C have a common element j and $(B,C) \in {K_j}^2$. Hence,

$$R = {K_1}^2 \cup {K_2}^2 \cup \cdots \cup {K_n}^2.$$

Clearly, $\wp(A) \setminus \{\emptyset\} = K_1 \cup K_2 \cup ... \cup K_n$, that is, $\mathcal{H}$ is a covering of U. This covering is irredundant, because if we omit K_j, then the set $\{j\}$ cannot be covered. Because $\wp(U)$ is finite, by Proposition 3.6 this means that $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$ are finite Boolean lattices.

For a tolerance R, an *R-path* is a sequence $a_0, a_1, \dots, a_n$ of distinct elements of U such that $a_i \, R \, a_{i+1}$ for all $0 \leq i \leq n-1$. The *length* of a path is the number of elements in the sequence minus one. Note that each point of U forms a path of length zero. We denote by $\overline{R}$ the transitive closure of R, that is,

$$\overline{R} = R \cup R^2 \cup R^3 \cup \cdots \cup R^n \cup \cdots \ .$$

Then, $\overline{R}$ is the smallest equivalence containing R. Note that for any $X \subseteq U$, the upper $\overline{R}$-approximation $\overline{R}(X)$ of X consists of the elements that are connected to at least one element in X by an R-path. We denote $\overline{R}(X)$ simply by $\overline{X}$.

For any $a \in \overline{X}$, we define the *distance of a from X*, denoted $\delta(a,X)$, as the minimal length of an R-path connecting a at least to one element in X. Note that if $a \in X$, then $\delta(a,X) = 0$. For any $n \geq 0$, let us define the set

$$X^n = \{a \in U \mid \delta(a,X) = n\}.$$

The above means that $\overline{X} = \bigcup_{n \geq 0} X^n$. In addition, we denote:

$$X_{\text{even}} = X^0 \cup X^2 \cup X^4 \cup \cdots \cup X^{2k} \cup \cdots$$
$$X_{\text{odd}} = X^1 \cup X^3 \cup X^5 \cup \cdots \cup X^{2k+1} \cup \cdots$$

Our the next lemma presents some properties of X_{even} and X_{odd} that will be needed in the proofs of the next section.

Lemma 3.8. *If ρ be a tolerance on U, then the following assertions hold for all $X \subseteq U$:*
(a) *$\overline{X}$ is a disjoint union of X_{even} and X_{odd}.*
(b) *$X \subseteq X_{\text{even}}$ and $\rho(X) \setminus X \subseteq X_{\text{odd}}$.*
(c) *$X_{\text{odd}} \subseteq \rho(X_{\text{even}}) = \overline{X}$ and $\overline{X} \setminus X \subseteq \rho(X_{\text{odd}})$.*
(d) *$X \subseteq \rho(\overline{X} \setminus X)$ implies $\rho(X_{\text{odd}}) = \rho(X_{\text{even}}) = \overline{X}$.*

Proof. Claim (a) is obvious.

(b) Clearly $X = X^0 \subseteq X_{\text{even}}$, and $\rho(X) = X \cup X^1$ implies $\rho(X) \setminus X = X^1 \subseteq X_{\text{odd}}$.

(c) Suppose that $a \in X_{\text{odd}}$. Then, there is a ρ-path $(a_0, \dots, a_{2k}, a_{2k+1})$ of length $2k+1$ such that $a_0 \in X$ and $a_{2k+1} = a$. So, $(a_0, \dots, a_{2k})$ is a ρ-path of length $2k$ and thus $a_{2k} \in X_{\text{even}}$. Now $(a, a_{2k}) \in \rho$ gives $a \in \rho(X_{\text{even}})$. In addition, $\overline{X} = X_{\text{odd}} \cup X_{\text{even}} \subseteq \rho(X_{\text{even}}) \subseteq \overline{X}$. For the other part, let $a \in \overline{X} \setminus X$. Then, we have $a \in X^n$ for some $n \geq 1$. If n is an odd number, then $a \in X_{\text{odd}} \subseteq \rho(X_{\text{odd}})$. If n is even, then $n-1$ is odd and $a \in \rho(X^{n-1}) \subseteq \rho(X_{\text{odd}})$.

(d) Assume $X \subseteq \rho(\overline{X} \setminus X)$. Then, for each $x \in X$, there is $y \in \overline{X} \setminus X$ such that $x \, \rho \, y$. Since $y \in X^1 \subseteq X_{\text{odd}}$, we get $x \in \rho(X_{\text{odd}})$ and $X \subseteq \rho(X_{\text{odd}})$. Thus, $\overline{X} = X \cup (\overline{X} \setminus X) \subseteq \rho(X_{\text{odd}}) \subseteq \overline{X}$. $\square$

4. Lattice structures of rough sets determined by tolerances

Let R be a tolerance on U. We begin by considering the set $\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$ ordered coordinatewise by $\subseteq$. It is clear that $\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$ is a complete lattice such that

$$\bigwedge_{i\in I}(A_i, B_i) = \Big(\bigcap_{i\in I} A_i, \big(\bigcap_{i\in I} B_i\big)^{\blacktriangledown\blacktriangle}\Big) \tag{4.1}$$

and

$$\bigvee_{i\in I}(A_i, B_i) = \Big(\big(\bigcup_{i\in I} A_i,\big)^{\blacktriangle\blacktriangledown}, \bigcup_{i\in I} B_i\Big) \tag{4.2}$$

for all $(A_i, B_i)_{i\in I} \subseteq \wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$. Let us define a map $\sim$ on $\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$ by setting for any $(A, B) \in \wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$:

$$\sim(A, B) = (B^c, A^c). \tag{4.3}$$

The map $\sim$ can be viewed as a so-called *De Morgan operation*, because it satisfies for all $(A, B) \in \wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$,

$$\sim\sim(A, B) = (A, B)$$

and

$$(A_1, B_1) \leq (A_2, B_2) \text{ if and only if } \sim(A_1, B_1) \geq \sim(A_2, B_2).$$

In this work, lattices with a De Morgan operation are called *polarity lattices*.

As in the case of equivalences, the set of all rough sets is denoted by $RS = \{(X^{\blacktriangledown}, X^{\blacktriangle}) \mid X \subseteq U\}$. Obviously, $RS \subseteq \wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$ and RS is bounded with $(\emptyset, \emptyset)$ as the least element and (U, U) as the greatest element. It is known [9] that RS is self-dual by the map $\sim$, and for all $X \subseteq U$,

$$\sim(X^{\blacktriangledown}, X^{\blacktriangle}) = (X^{\blacktriangle c}, X^{\blacktriangledown c}) = (X^{c\blacktriangle}, X^{c\blacktriangledown}).$$

Recall that the map $X \mapsto X^{\blacktriangle\blacktriangledown}$ is a closure operator and $X \mapsto X^{\blacktriangledown\blacktriangle}$ is an interior operator (see Section 3). Our next lemma shows how RS can be also represented up to isomorphism as interior-closure pairs.

Lemma 4.1. *If R is a tolerance, then $RS \cong \{(X^{\blacktriangledown\blacktriangle}, X^{\blacktriangle\blacktriangledown}) \mid X \subseteq U\}$.*

Proof. We show that the map $\varphi\colon (X^{\blacktriangledown}, X^{\blacktriangle}) \mapsto (X^{\blacktriangledown\blacktriangle}, X^{\blacktriangle\blacktriangledown})$ is an order-isomorphism. If $(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle})$, then $X^{\blacktriangledown} \subseteq Y^{\blacktriangledown}$ implies $X^{\blacktriangledown\blacktriangle} \subseteq Y^{\blacktriangledown\blacktriangle}$. Similarly, $X^{\blacktriangle} \subseteq Y^{\blacktriangle}$ gives $X^{\blacktriangle\blacktriangledown} \subseteq Y^{\blacktriangle\blacktriangledown}$. Thus, $(X^{\blacktriangledown\blacktriangle}, X^{\blacktriangle\blacktriangledown}) \leq (Y^{\blacktriangledown\blacktriangle}, Y^{\blacktriangle\blacktriangledown})$.

On the other hand, if $(X^{\blacktriangledown\blacktriangle}, X^{\blacktriangle\blacktriangledown}) \leq (Y^{\blacktriangledown\blacktriangle}, Y^{\blacktriangle\blacktriangledown})$, then $X^{\blacktriangledown\blacktriangle} \subseteq Y^{\blacktriangledown\blacktriangle}$ implies $X^{\blacktriangledown} = X^{\blacktriangledown\blacktriangle\blacktriangledown} \subseteq Y^{\blacktriangledown\blacktriangle\blacktriangledown} = Y^{\blacktriangledown}$, and from $X^{\blacktriangle\blacktriangledown} \subseteq Y^{\blacktriangle\blacktriangledown}$ we get $X^{\blacktriangle} = X^{\blacktriangle\blacktriangledown\blacktriangle} \subseteq Y^{\blacktriangle\blacktriangledown\blacktriangle} = Y^{\blacktriangle}$. So, $(X^{\blacktriangledown}, X^{\blacktriangle}) \leq (Y^{\blacktriangledown}, Y^{\blacktriangle})$.

Thus, φ is an order-embedding. The map φ is surjective, because any pair $(X^{\blacktriangledown\blacktriangle}, X^{\blacktriangle\blacktriangledown})$ is the image of $(X^{\blacktriangledown}, X^{\blacktriangle}) \in RS$. $\square$

However, not any pair of interior and closure operators defines up to isomorphism the same structure as rough sets determined by tolerances. We present the following characterization of rough sets in terms of interior and closure operators.

Proposition 4.2. *Let $\mathcal{I}$ and $\mathcal{C}$ be lattice-theoretical interior and closure operators on the set U. Then, there exists a tolerance on U such that $RS \cong \{(\mathcal{I}(X), \mathcal{C}(X)) \mid X \subseteq U\}$ if and only if there exists a Galois connection (F, G) on $\wp(U)$ such that $\mathcal{C} = G \circ F$, $\mathcal{I} = F \circ G$ and the following conditions hold for all $x, y \in U$:*

(i) $x \in F(\{x\})$;
(ii) $x \in F(\{y\})$ *implies* $y \in F(\{x\})$.

Proof. ($\Rightarrow$) Let R be a tolerance on U. We denote the closure operator $X \mapsto X^{\blacktriangle\blacktriangledown}$ by $\mathcal{C}$ and the interior operator $X \mapsto X^{\blacktriangledown\blacktriangle}$ by $\mathcal{I}$. Then, RS is order-isomorphic to $\{(\mathcal{I}(X), \mathcal{C}(X)) \mid X \subseteq U\}$ by Lemma 4.1. Because $R(x) = \{x\}^{\blacktriangle}$ for all $x \in U$, conditions (i) and (ii) hold.

($\Leftarrow$) Suppose that (F, G) is a Galois connection satisfying $\mathcal{C} = G \circ F$, $\mathcal{I} = F \circ G$, and that conditions (i) and (ii) hold for F. Let us define a relation R by setting $R(x) = F(\{x\})$. As in the proof of Proposition 3.1, we can see that R is a tolerance on U such that $X^{\blacktriangle} = F(X)$ and $X^{\blacktriangledown} = G(X)$ for all $X \subseteq U$. It is now clear that for all $X \subseteq U$, $\mathcal{I}(X) = X^{\blacktriangledown\blacktriangle}$ and $\mathcal{C}(X) = X^{\blacktriangle\blacktriangledown}$, and $RS \cong \{(\mathcal{I}(X), \mathcal{C}(X)) \mid X \subseteq U\}$ follows from Lemma 4.1. $\square$

If $|U| \leq 4$, then RS is a lattice, but when $|U| \geq 5$, RS does not necessarily form a lattice, as can be seen in the following example; see [6, 7].

Example 4.3. Let R be a tolerance on $U = \{a, b, c, d, e\}$ such that $R(a) = \{a, b\}$, $R(b) = \{a, b, c\}$, $R(c) = \{b, c, d\}$, $R(d) = \{c, d, e\}$, and $R(e) = \{d, e\}$. The ordered set RS is depicted in Figure 1.

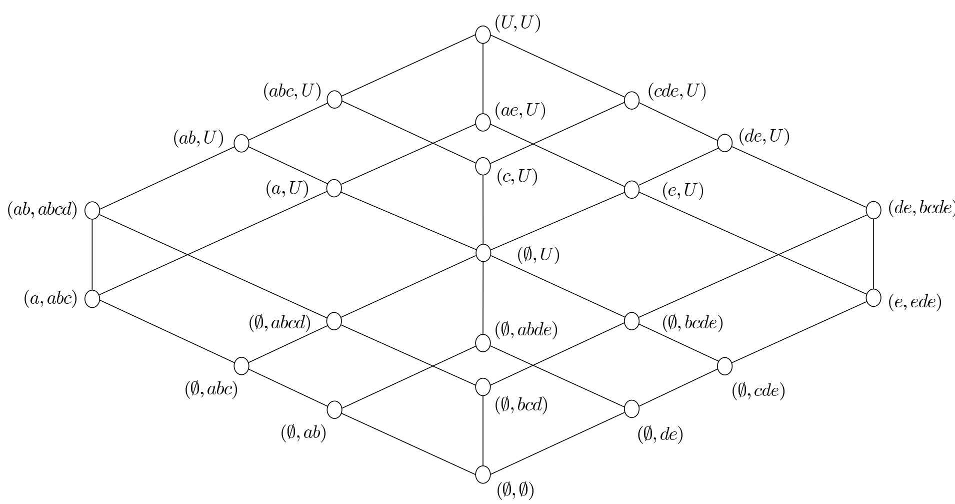

FIGURE 1.

For instance, the elements (a, abc) and $(\emptyset, abcd)$ do not have a least upper bound. Similarly, $(ab, abcd)$ and (a, U) do not have the greatest lower bound.

Next we consider completions of RS. Let us define the set

$$\mathcal{S} = \{x \in U \mid R(x) = \{x\}\}, \tag{4.4}$$

that is, $\mathcal{S}$ consists of such x's that $R(x)$ is a singleton. Then, $\{x\}^{\blacktriangledown} = \{x\}$ for all $x \in \mathcal{S}$ and $\mathcal{S}^{\blacktriangledown} = \mathcal{S} = \mathcal{S}^{\blacktriangle}$. Clearly, for any $X \subseteq U$ and $x \in \mathcal{S}$,

$$x \in X^{\blacktriangle} \iff R(x) \cap X \neq \emptyset \iff R(x) \subseteq X \iff x \in X^{\blacktriangledown}.$$

Hence, for all $x \in \mathcal{S}$, we have either $x \in X^{\blacktriangledown}$ or $x \in X^{\blacktriangle c}$. This means that $\mathcal{S} \subseteq X^{\blacktriangledown} \cup X^{\blacktriangle c}$ holds for any $X \subseteq U$. We define the set of pairs

$$\mathcal{I}(RS) = \{(A,B) \in \wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle} \mid A^{\blacktriangle} \subseteq B^{\blacktriangledown} \text{ and } \mathcal{S} \subseteq A \cup B^c\}.$$

Because $(X^{\blacktriangledown})^{\blacktriangle} \subseteq X \subseteq (X^{\blacktriangle})^{\blacktriangledown}$ and $\mathcal{S} \subseteq X^{\blacktriangledown} \cup X^{\blacktriangle c}$ for any $(X^{\blacktriangledown}, X^{\blacktriangle}) \in RS$, we have $RS \subseteq \mathcal{I}(RS)$.

The *Dedekind–MacNeille completion* of an ordered set P can be defined as the smallest complete lattice with P order-embedded in it (see [3], for example). In [19], D. Umadevi proved that for a reflexive relation R on U, the Dedekind–MacNeille completion of RS is

$$\mathcal{DM}(RS) = \{(A,B) \in \wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle} \mid A^{\blacktriangle\vartriangle} \subseteq B \text{ and } A \cap \mathcal{S} = B \cap \mathcal{S}\},$$

where $X^{\vartriangle}$ denotes the upper approximation of X defined in terms of the inverse R^{-1} of the relation R, that is, $X^{\vartriangle} = \{x \in U \mid R^{-1}(x) \cap X \neq \emptyset\}$. If R is a tolerance, we have $A^{\vartriangle} = A^{\blacktriangle}$ and $A^{\blacktriangle\vartriangle} \subseteq B \iff A^{\blacktriangle\blacktriangle} \subseteq B \iff A^{\blacktriangle} \subseteq B^{\blacktriangledown}$ for any $A, B \subseteq U$. Additionally,

$$\mathcal{S} \subseteq A \cup B^c \iff \mathcal{S} \cap (B \setminus A) = \emptyset \iff \mathcal{S} \cap B = \mathcal{S} \cap A. \tag{4.5}$$

Hence, for tolerances, we have $\mathcal{I}(RS) = \mathcal{DM}(RS)$, and consequently, $RS = \mathcal{I}(RS)$ holds whenever RS is a complete lattice. In [10], we proved that for any quasiorder R on U, RS is a complete, completely distributive lattice and

$$RS = \{(A,B) \in \wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle} \mid A \subseteq B \text{ and } \mathcal{S} \subseteq A \cup B^c\}.$$

Therefore, $\mathcal{I}(\mathcal{RS})$ can be called the *increasing representation of rough sets*.

A *complete subdirect product* $\mathcal{L}$ of an indexed family of complete lattices $\{L_i\}_{i \in I}$ is a complete sublattice of the direct product $\prod_{i\in I} L_i$ such that the canonical projections π_i are all surjective, that is, $\pi_i(\mathcal{L}) = L_i$. Note that the projections π_i are complete lattice homomorphisms, that is, they preserve all meets and joins.

Proposition 4.4. *Let R be a tolerance on U.*

(a) *$\mathcal{I}(RS)$ is a complete polarity sublattice of the polarity lattice $\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$.*
(b) *$\mathcal{I}(RS)$ is a complete subdirect product of $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$.*

Proof. (a) We first note that the map $\sim$ defined in (4.3) is a De Morgan operation on $\mathcal{I}(RS)$. If $(A,B) \in \mathcal{I}(RS)$, then $A^{\blacktriangle} \subseteq B^{\blacktriangledown}$ implies $B^{c\blacktriangle} = B^{\blacktriangledown c} \subseteq A^{\blacktriangle c} = A^{c\blacktriangledown}$. Additionally, $\mathcal{S} \subseteq A \cup B^c = B^c \cup (A^c)^c$. So, $\sim(A,B) = (B^c, A^c) \in \mathcal{I}(RS)$.

Let $\{(A_i,B_i)\}_{i\in I} \subseteq \mathcal{I}(RS)$. Its meet defined in $\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$ is

$$\bigwedge_{i\in I}(A_i,B_i) = \Big(\bigcap_{i\in I} A_i, \big(\bigcap_{i\in I} B_i\big)^{\blacktriangledown\blacktriangle}\Big).$$

We show that this meet is in $\mathcal{I}(RS)$. For all $i \in I$, we have $A_i{}^{\blacktriangle} \subseteq B_i{}^{\blacktriangledown}$. Thus, for all $i \in I$, we have $A_i \subseteq A_i{}^{\blacktriangle\blacktriangledown} \subseteq B_i{}^{\blacktriangledown\blacktriangledown}$, and $\bigcap_{i\in I} A_i \subseteq \bigcap_{i\in I} B_i^{\blacktriangledown\blacktriangledown} = (\bigcap_{i\in I} B_i)^{\blacktriangledown\blacktriangledown}$. This implies $(\bigcap_{i\in I} A_i)^{\blacktriangle} \subseteq (\bigcap_{i\in I} B_i)^{\blacktriangledown\blacktriangledown\blacktriangle} \subseteq (\bigcap_{i\in I} B_i)^{\blacktriangledown} = \big((\bigcap_{i\in I} B_i)^{\blacktriangledown\blacktriangle}\big)^{\blacktriangledown}$. For the second part, assume that $\mathcal{S} \nsubseteq \bigcap_{i\in I} A_i \cup (\bigcap_{i\in I} B_i)^{\blacktriangledown\blacktriangle c}$. This means that there exists $x \in \mathcal{S}$ such that $x \notin \bigcap_{i\in I} A_i$ and $x \notin (\bigcap_{i\in I} B_i)^{\blacktriangledown\blacktriangle c}$. Therefore, there is $k \in I$ with $x \notin A_k$ and $x \in (\bigcap_{i\in I} B_i)^{\blacktriangledown\blacktriangle} \subseteq \bigcap_{i\in I} B_i \subseteq B_k$. We get $x \notin A_k \cup B_k{}^c$, contradicting our assumption $\mathcal{S} \subseteq A_i \cup B_i{}^c$ for all $i \in I$. Thus, $\mathcal{S} \subseteq \bigcap_{i\in I} A_i \cup (\bigcap_{i\in I} B_i)^{\blacktriangledown\blacktriangle c}$ must hold. Additionally, the join

$$\bigvee_{i\in I}(A_i,B_i) = \Big(\big(\bigcup_{i\in I} A_i\big)^{\blacktriangle\blacktriangledown}, \bigcup_{i\in} B_i\Big)$$

defined in $\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$ equals $\sim \bigwedge_{i \in I} \sim(A_i, B_i)$, which, by the above, belongs to $\mathcal{I}(RS)$. Thus, $\mathcal{I}(RS)$ is a complete sublattice of $\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$.

(b) The maps $\pi_1 \colon (X^{\blacktriangledown}, Y^{\blacktriangle}) \mapsto X^{\blacktriangledown}$ and $\pi_2 \colon (X^{\blacktriangledown}, Y^{\blacktriangle}) \mapsto Y^{\blacktriangle}$ are the canonical projections of the product $\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$. Obviously, their restrictions to $\mathcal{I}(\mathcal{RS})$ are surjective, because $RS \subseteq \mathcal{I}(RS)$. Combined with (a), this proves the claim. $\square$

Corollary 4.5. *RS is a complete lattice if and only if it is a complete subdirect product of the complete lattices $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$.*

Remark 4.6. The fact that for a tolerance R on U, $\mathcal{I}(RS)$ is the Dedekind–MacNeille completion of RS can be proved independently of [19] by showing that RS is both join-dense and meet-dense in $\mathcal{I}(RS)$. It is known that a complete lattice L is the Dedekind–MacNeille completion of an ordered subset P of L, whenever P is both join-dense and meet-dense in L, that is, every element of L can be represented as a join and a meet of some elements of P (see e.g. [3, Theorem 7.41]). In fact, one can show that for any pair $(A, B) \in \mathcal{I}(RS)$, we have

$$(A, B) = \bigvee \big(\{(R(x)^{\blacktriangledown}, R(x)^{\blacktriangle} \mid x \in A\} \cup \{(\emptyset, R(x)) \mid x \in B^{\blacktriangledown} \setminus A\}\big). \tag{4.6}$$

Trivially, the pairs $(R(x)^{\blacktriangledown}, R(x)^{\blacktriangle})$ are rough sets for every $x \in A$. If $x \in B^{\blacktriangledown} \setminus A$, then $x \notin \mathcal{S}$ by (4.5), and hence $\{x\}^{\blacktriangledown} = \emptyset$ and $(\{x\}^{\blacktriangledown}, \{x\}^{\blacktriangle}) = (\emptyset, R(x))$ is a rough set. Thus, (4.6) implies that RS is join-dense in $\mathcal{I}(RS)$. Because $\mathcal{I}(RS)$ and RS are self-dual by the map $\sim$, RS is also meet-dense in $\mathcal{I}(RS)$.

By Corollary 4.5, to show that RS is a complete lattice it is enough to prove that RS is a complete sublattice of $\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$. Additionally, since RS is a self-dual subset of the complete polarity lattice $\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangle}$, it suffices to find for any $\mathcal{H} \subseteq \wp(U)$ a set $Z \subseteq U$ such that

$$Z^{\blacktriangledown} = \bigcap_{X \in \mathcal{H}} X^{\blacktriangledown} \quad \text{and} \quad Z^{\blacktriangle} = \Big(\bigcap_{X \in \mathcal{H}} X^{\blacktriangle}\Big)^{\blacktriangledown\blacktriangle}. \tag{4.7}$$

Observe that

$$\Big(\bigcap_{X \in \mathcal{H}} X\Big)^{\blacktriangledown\blacktriangle} = \Big(\bigcap_{X \in \mathcal{H}} X^{\blacktriangledown}\Big)^{\blacktriangle} = Z^{\blacktriangledown\blacktriangle} \subseteq Z \subseteq Z^{\blacktriangle} = \Big(\bigcap_{X \in \mathcal{H}} X^{\blacktriangle}\Big)^{\blacktriangledown\blacktriangle}.$$

So, we have a lower bound and an upper bound for this Z. Especially, concerning Lemma 4.7, the interpretation is that

$$T = \Big(\bigcap_{X \in \mathcal{H}} X\Big)^{\blacktriangledown\blacktriangle} \quad \text{and} \quad Y = \Big(\bigcap_{X \in \mathcal{H}} X^{\blacktriangle}\Big)^{\blacktriangledown\blacktriangle}.$$

Lemma 4.7. *Let $Y, T \subseteq U$ be such that $Y \in \wp(U)^{\blacktriangle}$ and $T \subseteq Y^{\blacktriangledown}$. If $|R(x)| \geq 2$ for all $x \in Y \setminus T^{\blacktriangle}$, then there exists a set $S \subseteq Y^{\blacktriangledown} \setminus T$ such that $Y = S^{\blacktriangle} \cup T^{\blacktriangle}$ and $R(y) \nsubseteq S \cup T$ for all $y \in S$.*

Proof. Since $T \subseteq Y^{\blacktriangledown} \subseteq Y$, the set $Y \setminus T$ is a disjoint union of $Y \setminus Y^{\blacktriangledown}$ and $Y^{\blacktriangledown} \setminus T$, and hence

$$(Y \setminus T) \setminus (Y \setminus Y^{\blacktriangledown}) = Y^{\blacktriangledown} \setminus T. \tag{4.8}$$

Clearly, $T^{\blacktriangle} \subseteq Y^{\blacktriangledown\blacktriangle} = Y$, because $Y \in \wp(U)^{\blacktriangle}$. If $T^{\blacktriangle} = Y$, then our assertion is satisfied trivially with $S = \emptyset$. Thus, we may suppose $T^{\blacktriangle} \subset Y$, which yields $Y \setminus T \neq \emptyset$, because $T \subseteq T^{\blacktriangle}$. Let ρ denote the restriction of R to the set $Y \setminus T$.

Then, ρ is a tolerance, and its transitive closure $\overline{\rho}$ is an equivalence on $Y \setminus T$. Now, consider the sets

$$A = \{y \in Y \setminus T \mid \overline{\rho}(y) \cap (Y \setminus Y^{\blacktriangledown}) \neq \emptyset\};$$
$$B = \{y \in Y \setminus T \mid \overline{\rho}(y) \cap (Y \setminus Y^{\blacktriangledown}) = \emptyset \text{ and } \overline{\rho}(y) \nsubseteq T^{\blacktriangle}\}.$$

Then, $(Y \setminus Y^{\blacktriangledown}) \cap B \subseteq A \cap B = \emptyset$, and by (4.8) we obtain

$$B \subseteq (Y \setminus T) \setminus (Y \setminus Y^{\blacktriangledown}) = Y^{\blacktriangledown} \setminus T.$$

We apply Lemma 3.8 with the tolerance ρ and the sets $U = Y \setminus T$, $X = Y \setminus Y^{\blacktriangledown}$, and $\overline{X} = \overline{\rho}(Y \setminus Y^{\blacktriangledown}) = A$. We obtain two disjoint sets $(Y \setminus Y^{\blacktriangledown})_{\text{odd}}$ and $(Y \setminus Y^{\blacktriangledown})_{\text{even}}$ such that $(Y \setminus Y^{\blacktriangledown})_{\text{odd}} \cup (Y \setminus Y^{\blacktriangledown})_{\text{even}} = A$, and

$$Y \setminus Y^{\blacktriangledown} \subseteq (Y \setminus Y^{\blacktriangledown})_{\text{even}}; \tag{4.9}$$
$$(Y \setminus Y^{\blacktriangledown})_{\text{odd}} \subseteq \rho((Y \setminus Y^{\blacktriangledown})_{\text{even}}); \tag{4.10}$$
$$A \setminus (Y \setminus Y^{\blacktriangledown}) \subseteq \rho((Y \setminus Y^{\blacktriangledown})_{\text{odd}}). \tag{4.11}$$

Next, let $\Pi = \{H_k \mid k \in K\}$ be the partition induced by the equivalence $\overline{\rho}$ on B. Note that $\overline{\rho}(B) = B$, because $x \in B$ and $\overline{\rho}(x) = \overline{\rho}(y)$ imply $y \in B$. For each $k \in K$, we may select an element $c_k \in H_k \subseteq B$ such that $c_k \notin T^{\blacktriangle}$. This is because $H_k = \overline{\rho}(b)$ for some $b \in B$ and B was defined so that $\overline{\rho}(b) \nsubseteq T^{\blacktriangle}$ for all $b \in B$. Denote the set of all these elements by C, that is,

$$C = \{c_k \mid k \in K\}.$$

Then, $C \subseteq B \setminus T^{\blacktriangle} \subseteq Y^{\blacktriangledown} \setminus T^{\blacktriangle}$. Observe that $C \subseteq \rho(B \setminus C)$. Indeed, $C \subseteq Y \setminus T^{\blacktriangle}$ yields $|R(x)| \geq 2$ for all $x \in C$ by our assumption. Thus, for each $x \in C$ there is an element $y \neq x$ with $x \, R \, y$. Then, $y \in C^{\blacktriangle} \subseteq Y^{\blacktriangledown\blacktriangle} \subseteq Y$, and $x \notin T^{\blacktriangle}$ yields $y \notin T$. Hence, $x, y \in Y \setminus T$ and $x \, \rho \, y$ holds also. This implies $y \in \overline{\rho}(x) \subseteq B$. Since $x \in C$ is the unique element picked from the set $\overline{\rho}(x) \in \Pi$, $y \neq x$ implies $y \in B \setminus C$. This proves $C \subseteq \rho(B \setminus C)$.

By applying Lemma 3.8 again with the tolerance ρ and the sets $X = C \subseteq Y \setminus T$ and $\overline{X} = B$, we obtain two disjoint sets C_{odd} and C_{even} such that $C_{\text{odd}} \cup C_{\text{even}} = B$. Because $C \subseteq \rho(B \setminus C)$, we have

$$\rho(C_{\text{odd}}) = \rho(C_{\text{even}}) = B. \tag{4.12}$$

Finally, consider the set

$$S = (Y \setminus Y^{\blacktriangledown})_{\text{odd}} \cup C_{\text{odd}}.$$

We prove that S has the required properties, that is, (i) $S \subseteq Y^{\blacktriangledown} \setminus T$, (ii) $S^{\blacktriangle} \cup T^{\blacktriangle} = Y$, and (iii) $R(y) \nsubseteq S \cup T$ for all $y \in S$.

(i) Obviously, $C_{\text{odd}} \subseteq B \subseteq Y^{\blacktriangledown} \setminus T$ and $(Y \setminus Y^{\blacktriangledown})_{\text{odd}} \subseteq A \subseteq Y \setminus T$. By (4.9),

$$(Y \setminus Y^{\blacktriangledown})_{\text{odd}} \cap (Y \setminus Y^{\blacktriangledown}) \subseteq (Y \setminus Y^{\blacktriangledown})_{\text{odd}} \cap (Y \setminus Y^{\blacktriangledown})_{\text{even}} = \emptyset.$$

Then, (4.8) yields $(Y \setminus Y^{\blacktriangledown})_{\text{odd}} \subseteq Y^{\blacktriangledown} \setminus T$, and $S = (Y \setminus Y^{\blacktriangledown})_{\text{odd}} \cup C_{\text{odd}} \subseteq Y^{\blacktriangledown} \setminus T$.

(ii) Since $S, T \subseteq Y^{\blacktriangledown}$, we have $S^{\blacktriangle} \cup T^{\blacktriangle} = (S \cup T)^{\blacktriangle} \subseteq Y^{\blacktriangledown\blacktriangle} \subseteq Y$. For the other direction, assume $x \in Y \setminus T^{\blacktriangle} \subseteq Y \setminus T$. Then, either $\overline{\rho}(x) \cap (Y \setminus Y^{\blacktriangledown}) = \emptyset$ or $\overline{\rho}(x) \cap (Y \setminus Y^{\blacktriangledown}) \neq \emptyset$ holds. Because $x \notin T^{\blacktriangle}$, we have $\overline{\rho}(x) \nsubseteq T^{\blacktriangle}$. Therefore, if $\overline{\rho}(x) \cap (Y \setminus Y^{\blacktriangledown}) = \emptyset$, then $x \in B$. Since $B = \rho(C_{\text{odd}})$ by (4.12) and $\rho(C_{\text{odd}}) \subseteq S \subseteq S^{\blacktriangle}$, we get $x \in S^{\blacktriangle}$. If $\overline{\rho}(x) \cap (Y \setminus Y^{\blacktriangledown}) \neq \emptyset$, then $x \in A$, and either $x \in Y \setminus Y^{\blacktriangledown}$ or $x \notin Y \setminus Y^{\blacktriangledown}$.

If $x \in Y \setminus Y^{\blacktriangledown}$, then $x \in Y = Y^{\blacktriangledown\blacktriangle}$ yields that $x \, R \, y$ for some $y \in Y^{\blacktriangledown}$. Since $x \notin T^{\blacktriangle}$, we have $y \notin T$ and $y \in Y \setminus T$. By $x, y \in Y \setminus T$, we obtain $x \, \rho \, y$. The

facts $x \in Y \setminus Y^{\blacktriangledown}$ and $y \notin Y \setminus Y^{\blacktriangledown}$ imply $y \in (Y \setminus Y^{\blacktriangledown})^1 \subseteq (Y \setminus Y^{\blacktriangledown})_{\text{odd}}$, which gives $x \in \rho((Y \setminus Y^{\blacktriangledown})_{\text{odd}}) \subseteq ((Y \setminus Y^{\blacktriangledown})_{\text{odd}})^{\blacktriangle} \subseteq S^{\blacktriangle}$. If $x \notin Y \setminus Y^{\blacktriangledown}$, then

$$x \in A \setminus (Y \setminus Y^{\blacktriangledown}) \subseteq \rho((Y \setminus Y^{\blacktriangledown})_{\text{odd}}) \subseteq ((Y \setminus Y^{\blacktriangledown})_{\text{odd}})^{\blacktriangle} \subseteq S^{\blacktriangle}$$

by (4.11). As we obtained $x \in S^{\blacktriangle}$ in all possible cases, $Y \subseteq S^{\blacktriangle} \cup T^{\blacktriangle}$ holds.

(iii) Let $x \in S = (Y \setminus Y^{\blacktriangledown})_{\text{odd}} \cup C_{\text{odd}}$. Since $(Y \setminus Y^{\blacktriangledown})_{\text{odd}} \cap C_{\text{odd}} \subseteq A \cap B = \emptyset$, either $x \in (Y \setminus Y^{\blacktriangledown})_{\text{odd}}$ or $x \in C_{\text{odd}}$. In the first case, $x \in \rho((Y \setminus Y^{\blacktriangledown})_{\text{even}})$ by (4.10). Hence, there is an element $y \in (Y \setminus Y^{\blacktriangledown})_{\text{even}}$ with $(x, y) \in \rho \subseteq R$, that is, $y \in R(x)$. The inclusion $(Y \setminus Y^{\blacktriangledown})_{\text{even}} \subseteq A \subseteq Y \setminus T$ gives $y \notin T$. Because

$$\begin{aligned} S \cap (Y \setminus Y^{\blacktriangledown})_{\text{even}} &= ((Y \setminus Y^{\blacktriangledown})_{\text{odd}} \cap (Y \setminus Y^{\blacktriangledown})_{\text{even}}) \cup (C_{\text{odd}} \cap (Y \setminus Y^{\blacktriangledown})_{\text{even}}) \\ &\subseteq \emptyset \cup (B \cap A) = \emptyset, \end{aligned}$$

we get $y \notin S$. Thus, $R(x) \nsubseteq S \cup T$. If $x \in C_{\text{odd}}$, then $x \in B = \rho(C_{\text{even}})$ by (4.12). Hence, there is $y \in C_{\text{even}}$ with $(x, y) \in \rho \subseteq R$, that is, $y \in R(x)$. Since $C_{\text{even}} \subseteq B \subseteq Y \setminus T$, we get $y \notin T$. Clearly, $y \notin C_{\text{odd}}$ and $C_{\text{even}} \cap (Y \setminus Y^{\blacktriangledown})_{\text{odd}} \subseteq B \cap A = \emptyset$ implies $y \notin (Y \setminus Y^{\blacktriangledown})_{\text{odd}}$. Hence, $y \notin C_{\text{odd}} \cup (Y \setminus Y^{\blacktriangledown})_{\text{odd}} = S$, and so $R(x) \nsubseteq S \cup T$. $\square$

An element x of a complete lattice L is said to be *compact* if for every subset S of L, $x \le \bigvee S$ implies $x \le \bigvee F$ for some finite subset F of S. A complete lattice is *algebraic* if its every element can be given as a join of compact elements.

Theorem 4.8. *Let R be a tolerance on U. Then RS is an algebraic completely distributive lattice if and only if R is induced by an irredundant covering of U.*

Proof. Suppose that RS is an algebraic completely distributive lattice. Then, RS is a complete lattice and, by Corollary 4.5, it is a complete subdirect product of the lattices $\wp(U)^{\blacktriangledown}$ and $\wp(U)^{\blacktriangle}$. Since $\wp(U)^{\blacktriangledown}$ is the image of RS under the complete lattice-homomorphism π_1, the lattice $\wp(U)^{\blacktriangledown}$ is also completely distributive. Hence, according to Proposition 3.4, R is induced by an irredundant covering of U.

Conversely, let R be a tolerance induced by an irredundant covering of U. Then R satisfies also condition (b) of Lemma 3.3. First, we show that RS is a complete lattice. Let $\mathcal{H} \subseteq \wp(U)$. By (4.7), it is enough to show that there exists a set $Z \subseteq U$ such that

$$(Z^{\blacktriangledown}, Z^{\blacktriangle}) = \Big(\bigcap_{X \in \mathcal{H}} X^{\blacktriangledown}, (\bigcap_{X \in \mathcal{H}} X^{\blacktriangle})^{\blacktriangledown\blacktriangle} \Big). \tag{4.13}$$

Let us first set

$$T = (\bigcap_{X \in \mathcal{H}} X)^{\blacktriangledown\blacktriangle} \quad \text{and} \quad Y = (\bigcap_{X \in \mathcal{H}} X^{\blacktriangle})^{\blacktriangledown\blacktriangle}. \tag{4.14}$$

Using the properties of $^{\blacktriangle}$ and $^{\blacktriangledown}$, it is clear that $T = T^{\blacktriangledown\blacktriangle}$, $Y = Y^{\blacktriangledown\blacktriangle}$, and

$$T \subseteq \bigcap_{X \in \mathcal{H}} X \subseteq \bigcap_{X \in \mathcal{H}} X^{\blacktriangle\blacktriangledown} = \Big(\bigcap_{X \in \mathcal{H}} X^{\blacktriangle} \Big)^{\blacktriangledown} = \Big(\bigcap_{X \in \mathcal{H}} X^{\blacktriangle} \Big)^{\blacktriangledown\blacktriangle\blacktriangledown} = Y^{\blacktriangledown}.$$

In addition, $T^{\blacktriangledown} = \big(\bigcap_{X \in \mathcal{H}} X\big)^{\blacktriangledown}$ and $T^{\blacktriangle} \subseteq Y^{\blacktriangledown\blacktriangle} = Y$. Suppose that $x \in Y \setminus T^{\blacktriangle}$ and $|R(x)| = 1$. Since $R(x) = \{x\}$, $x \in Y$ implies that $x \in X$ for all $X \in \mathcal{H}$, from which we get $x \in T^{\blacktriangle}$, a contradiction. Thus, we must have $|R(x)| \ge 2$ for all $x \in Y \setminus T^{\blacktriangle}$. Now, we may apply Lemma 4.7 with the sets T and Y, and this yields that there exists a set

$$S \subseteq Y^{\blacktriangledown} \setminus T \text{ with } S^{\blacktriangle} \cup T^{\blacktriangle} = Y \text{ and } R(y) \nsubseteq S \cup T \text{ for all } y \in S. \tag{4.15}$$

Let us define the set

$$V = \{v \in T \mid R(v) \nsubseteq T \text{ and } R(v) \subseteq S \cup T\}.$$

First, we prove that if $V = \emptyset$, the set $Z = S \cup T$ satisfies (4.13). Since $^\blacktriangle$ distributes over unions, we have $Z^\blacktriangle = S^\blacktriangle \cup T^\blacktriangle = Y$. Trivially, $T^\blacktriangledown \subseteq Z^\blacktriangledown$. On the other hand, if $z \in Z^\blacktriangledown$, then $R(z) \subseteq S \cup T$. We have $R(z) \subseteq T$, because $R(z) \nsubseteq T$ implies $z \in V$, but this is impossible because $V = \emptyset$. Thus, $Z^\blacktriangledown = T^\blacktriangledown$ and (4.13) holds.

Now we prove that condition (b) of Lemma 3.3 implies $V = \emptyset$, which by previous observation yields that RS is a complete lattice. Suppose $V \neq \emptyset$. Then, there exists $v \in T$ such that $R(v) \nsubseteq T$ and $R(v) \subseteq S \cup T$. This also means that there is an element $s \in R(v)$ with $s \in S \setminus T$. As $v \, R \, s$, by Lemma 3.3 there exists $c_v \in R(c_v) \subseteq R(v) \cap R(s)$ such that $R(c_v) \subseteq R(x)$ for all $x \, R \, c_v$. Then $c_v \, R \, s$, and in particular, we have

$$c_v \in R(c_v) \subseteq R(v) \subseteq S \cup T. \tag{4.16}$$

Observe that $c_v \notin T$, because $c_v \in T = T^{\blacktriangledown\blacktriangle}$ means that there is $a \in T^\blacktriangledown$ with $a \, R \, c_v$. Since $R(c_v) \subseteq R(x)$ for all $x \, R \, c_v$, we get $s \in R(c_v) \subseteq R(a) \subseteq T^{\blacktriangledown\blacktriangle} \subseteq T$, but this is not possible because $s \in S \setminus T$. Therefore, $c_v \in S$, and we have $R(c_v) \nsubseteq S \cup T$ by Lemma 4.7. But this contradicts (4.16), and we deduce $V = \emptyset$, which, as we have already noted, implies that RS is a complete lattice.

Finally, since RS is a complete lattice, it is isomorphic to a complete subdirect product of $\wp(U)^\blacktriangledown$ and $\wp(U)^\blacktriangle$ by Corollary 4.5. Since R is a tolerance induced by an irredundant covering, in view of Proposition 3.6, $\wp(U)^\blacktriangledown$ and $\wp(U)^\blacktriangle$ are complete atomistic Boolean lattices. Thus, they are completely distributive and algebraic, too. Hence, RS, as a complete subdirect product of two completely distributive algebraic lattices is also completely distributive and algebraic. $\square$

An *Alexandrov topology* is a topology $\mathcal{T}$ that contains all arbitrary intersections of its members. Alexandrov topologies are also called *complete rings of sets*. It is known that a lattice L is isomorphic to an Alexandrov topology if and only if L is completely distributive and algebraic (see e.g. [3]). By Theorem 4.8, RS is isomorphic to some Alexandrov topology whenever R is induced by an irredundant covering of U.

A *Heyting algebra* L is a bounded distributive lattice such that for all $a, b \in L$, there is a greatest element x of L such that $a \wedge x \leq b$. This element is the *relative pseudocomplement* of a with respect to b, and is denoted $a \Rightarrow b$. It is well known that any completely distributive lattice L is a Heyting algebra such that the relative pseudocomplement is defined as

$$x \Rightarrow y = \bigvee \big\{z \in L \mid z \wedge x \leq y\big\}. \tag{4.17}$$

Therefore, if R is a tolerance induced by an irredundant covering of U, then RS is a Heyting algebra.

A *Kleene algebra* is a structure $\mathbb{A} = (A, \vee, \wedge, \sim, 0, 1)$ such that A is a bounded distributive lattice and for all $a, b \in A$:

(K1) $\sim\sim a = a$,
(K2) $a \leq b$ if and only if $\sim b \leq \sim a$,
(K3) $a \wedge \sim a \leq b \vee \sim b$.

According to R. Cignoli [1], a *quasi-Nelson algebra* is a Kleene algebra $\mathbb{A}$ such that for each pair a and b of its elements, the relative pseudocomplement $a \Rightarrow ({\sim}a \vee b)$ exists. In quasi-Nelson algebras, $a \Rightarrow ({\sim}a \vee b)$ is denoted simply by $a \to b$ and this is called the *weak relative pseudocomplement* of a with respect to b. Obviously, each Kleene algebra such that its underlying lattice forms a Heyting algebra is a quasi-Nelson algebra. A *Nelson algebra* is a quasi-Nelson algebra $(A, \vee, \wedge, \to, \sim, 0, 1)$ satisfying the equation

$$(a \wedge b) \to c = a \to (b \to c).$$

In the case of quasiorders, it is shown by J. Järvinen, S. Radeleczki, and L. Veres [12] that RS forms a complete sublattice of $\wp(U) \times \wp(U)$ ordered by the coordinatewise set-inclusion relation. In addition, we have proved in [11] that RS determines a Nelson algebra.

Let the operation $\sim$ on RS be defined as in (4.3).

Proposition 4.9. *Let R be a tolerance induced by an irredundant covering of U. Then, the algebra*

$$\mathbb{RS} = (RS, \cup, \cap, \sim, (\emptyset, \emptyset), (U, U))$$

is a quasi-Nelson algebra.

Proof. If R is a tolerance induced by an irredundant covering of U, then by Theorem 4.8, RS is a complete distributive lattice bounded by $(\emptyset, \emptyset)$ and (U, U). As we have already noted, conditions (K1) and (K2) are satisfied. Let $\mathcal{A}(X) = (X^{\blacktriangledown}, X^{\blacktriangle})$ and $\mathcal{A}(Y) = (Y^{\blacktriangledown}, Y^{\blacktriangle})$ be in RS. Then,

$$\mathcal{A}(X) \wedge {\sim}\mathcal{A}(X) = (X^{\blacktriangledown} \cap X^{c\blacktriangledown}, (X^{\blacktriangle} \cap X^{c\blacktriangle})^{\blacktriangledown\blacktriangle}) = (\emptyset, (X^{\blacktriangle} \setminus X^{\blacktriangledown})^{\blacktriangledown\blacktriangle}), \text{ and}$$
$$\mathcal{A}(Y) \vee {\sim}\mathcal{A}(Y) = ((Y^{\blacktriangledown} \cup Y^{c\blacktriangledown})^{\blacktriangle\blacktriangledown}), Y^{\blacktriangle} \cup Y^{c\blacktriangle}) = ((Y^{\blacktriangledown\blacktriangle} \cup Y^{c\blacktriangledown\blacktriangle})^{\blacktriangledown}, U).$$

Hence, $\mathcal{A}(X) \wedge {\sim}\mathcal{A}(X) \leq \mathcal{A}(Y) \vee {\sim}\mathcal{A}(Y)$, and condition (K3) holds also.

Since RS is a Heyting algebra when R is a tolerance induced by an irredundant covering of U, the Kleene algebra $\mathbb{RS}$ is a quasi-Nelson algebra. □

Note that (K3) has nothing to do with distributivity, so if RS is a lattice, $\sim$ satisfies conditions (K1)–(K3), and even R is induced by an irredundant covering of U, the algebra $\mathbb{RS}$ does not necessarily form a Nelson algebra. For instance, if R is a tolerance on $U = \{a, b, c\}$ such that $R(a) = \{a, b\}$, $R(b) = U$, and $R(c) = \{b, c\}$, the quasi-Nelson algebra $\mathbb{RS}$ is not a Nelson algebra.

For a tolerance R on U and any $X \subseteq U$, we denote the restriction of R to X by R_X and by RS_X the set of all rough sets determined by the relation R_X on X. It is clear that for an equivalence R, the relation R_X is an equivalence and RS_X is a lattice for all $X \subseteq U$. Similar observation holds for quasiorders.

Let us introduce the following condition related to R-paths:

(C) For any R-path $(a_0, \ldots, a_4)$ of length 4, there exists $0 \leq i, j \leq 4$ such that $|i - j| \geq 2$ and $a_i \, R \, a_j$.

Lemma 4.10. *Let R be a tolerance on U. If RS_X is a lattice for all $X \subseteq U$ with $|X| = 5$, then R satisfies condition* (C).

Proof. Suppose R does not satisfy (C). Then, there exists an R-path $(a_0, \ldots, a_4)$ such that $a_i \, R \, a_j$ if and only if $|i - j| \leq 1$. Let us choose $X = \{a_0, \ldots, a_4\}$. Then $|X| = 5$ and the situation is exactly as in Example 4.3, that is, RS_X is not a lattice □

Now, we present our second main result.

Theorem 4.11. *If R is a tolerance satisfying* (C)*, then RS is a complete lattice.*

Proof. Let $\mathcal{H} \subseteq \wp(A)$. As in the proof of Theorem 4.8, we need to find a set $Z \subseteq U$ such that $Z^\blacktriangledown = \bigcap_{X \in \mathcal{H}} X^\blacktriangledown$ and $Z^\blacktriangle = (\bigcap_{X \in \mathcal{H}} X^\blacktriangle)^{\blacktriangledown\blacktriangle}$. Let us form now the sets T, Y, S, and V exactly as in the proof of Theorem 4.8, meaning that (4.14) and (4.15) hold. Recall that

$$V = \{v \in T \mid R(v) \nsubseteq T \text{ and } R(v) \subseteq S \cup T\}.$$

According to the proof of Theorem 4.8, $V = \emptyset$ implies that RS is a complete lattice, hence we may assume $V \neq \emptyset$. Now, for each $v \in V$, we can choose an element $q_v \in R(v)$ such that $q_v \in T^\blacktriangle \setminus T$. Denote by Q the set of these selected elements, that is, $Q = \{q_v \mid v \in V\}$. Then for all $v \in V$, $q_v \in T^\blacktriangle$, $q_v \notin T$, and $q_v \in R(v) \subseteq S \cup T$ give $q_v \in (S \setminus T) \cap T^\blacktriangle$, and so

$$Q \subseteq (S \setminus T) \cap T^\blacktriangle. \tag{4.18}$$

Next, we define a set $P \subseteq Y$ by setting

$$P = \{p \in Y \setminus ((S \setminus Q)^\blacktriangle \cup T^\blacktriangle) \mid R(p) \subseteq Q^\blacktriangle\}. \tag{4.19}$$

Then $P \subseteq Q^{\blacktriangle\blacktriangledown} \subseteq Q^\blacktriangle$, because for each $p \in P$, $R(p) \subseteq Q^\blacktriangle$. In addition,

$$P \cap ((S \setminus Q)^\blacktriangle \cup T^\blacktriangle) = P \cap ((S \setminus Q) \cup T)^\blacktriangle = \emptyset,$$

that is,

$$(\forall p \in P)\, R(p) \cap \big((S \setminus Q) \cup T\big) = \emptyset. \tag{4.20}$$

We have $Q \subseteq T^\blacktriangle$ by (4.18), which gives $P \cap Q \subseteq P \cap \big((S \setminus Q)^\blacktriangle \cup T^\blacktriangle\big) = \emptyset$.

Let us now define the set

$$Z = (S \setminus Q) \cup T \cup P.$$

We will prove that

$$Z^\blacktriangledown = \Big(\bigcap_{X \in \mathcal{H}} X\Big)^\blacktriangledown = T^\blacktriangledown \quad \text{and} \quad Z^\blacktriangle = \Big(\bigcap_{X \in \mathcal{H}} X^\blacktriangle\Big)^{\blacktriangledown\blacktriangle} = Y.$$

Trivially, $T^\blacktriangledown \subseteq Z^\blacktriangledown$. To prove $Z^\blacktriangledown \subseteq T^\blacktriangledown$, let $z \in Z^\blacktriangledown$. Then,

$$z \in R(z) \subseteq Z = (S \setminus Q) \cup T \cup P, \tag{4.21}$$

and we have $z \in S \setminus Q$, or $z \in T$, or $z \in P$. We first show that $z \in T$.

If $z \in P$, then $R(z) \cap \big((S \setminus Q) \cup T\big) = \emptyset$ by (4.20), and (4.21) gives $R(z) \subseteq P$. From this we obtain $R(z) \cap Q \subseteq P \cap Q = \emptyset$. On the other hand, $z \in P \subseteq Q^\blacktriangle$ yields $R(z) \cap Q \neq \emptyset$, a contradiction. Similarly, $z \in S \setminus Q$ gives $R(z) \subseteq (S \setminus Q)^\blacktriangle$. Then, $P \cap ((S \setminus Q)^\blacktriangle \cup T^\blacktriangle) = \emptyset$ implies $P \cap R(z) = \emptyset$ and we must have $R(z) \subseteq (S \setminus Q) \cup T \subseteq S \cup T$. Since $z \in S$, this contradicts $R(z) \nsubseteq S \cup T$ following from (4.15). Hence, the only possibility left is $z \in T$.

Next, we prove $z \in T^\blacktriangledown$. Suppose, by the way of contradiction, that $R(z) \nsubseteq T$. Since $R(z) \subseteq T^\blacktriangle$ and, by (4.19), $P \cap T^\blacktriangle = \emptyset$, we obtain $R(z) \cap P \subseteq T^\blacktriangle \cap P = \emptyset$, which implies $R(z) \subseteq (S \setminus Q) \cup T \subseteq S \cup T$. This means that $z \in V$. So, there exists an element $q_z \in Q$ with $q_z \in R(z)$. By the definition of Q, $q_z \notin T$. So, we have $q_z \notin (S \setminus Q) \cup T$, which contradicts $R(z) \subseteq (S \setminus Q) \cup T$. Therefore, we have now proved $R(z) \subseteq T$, that is, $z \in T^\blacktriangledown$.

To complete our proof, we need to show that $Z^\blacktriangle = Y$. Recall that $Y = S^\blacktriangle \cup T^\blacktriangle$ by (4.15). By the definition of Z, we have $Z^\blacktriangle = (S \setminus Q)^\blacktriangle \cup T^\blacktriangle \cup P^\blacktriangle$. In view of (4.19),

$R(p) \subseteq Q^{\blacktriangle}$ for all $p \in P$, which implies $P^{\blacktriangle} = \bigcup_{p \in P} R(p) \subseteq Q^{\blacktriangle} \subseteq S^{\blacktriangle}$, because $Q \subseteq (S \setminus T) \cap T^{\blacktriangle} \subseteq S$ holds by (4.18). Hence, we have $Z^{\blacktriangle} \subseteq (S \setminus Q)^{\blacktriangle} \cup T^{\blacktriangle} \cup S^{\blacktriangle} = T^{\blacktriangle} \cup S^{\blacktriangle}$. We show $Z^{\blacktriangle} = S^{\blacktriangle} \cup T^{\blacktriangle}$ by proving $(S^{\blacktriangle} \cup T^{\blacktriangle}) \setminus Z^{\blacktriangle} = \emptyset$.

Assume now that R satisfies (C) and suppose for contradiction that there exists an element $y \in (S^{\blacktriangle} \cup T^{\blacktriangle}) \setminus Z^{\blacktriangle} = Y \setminus Z^{\blacktriangle}$. Since $Q \subseteq S$, we have $S = Q \cup (S \setminus Q)$ and $y \in Q^{\blacktriangle} \cup (S \setminus Q)^{\blacktriangle} \cup T^{\blacktriangle}$. Because $y \notin Z^{\blacktriangle} = (S \setminus Q)^{\blacktriangle} \cup T^{\blacktriangle} \cup P^{\blacktriangle}$ yields $y \notin (S \setminus Q)^{\blacktriangle}$ and $y \notin T^{\blacktriangle}$, we must have $y \in Q^{\blacktriangle} \setminus T^{\blacktriangle}$. As $Q \subseteq T^{\blacktriangle}$, we get $y \in Q^{\blacktriangle} \setminus Q$. Therefore, there are $v \in V \subseteq T$ and $q_v \in Q$ such that $q_v \in R(v) \subseteq S \cup T$, $R(v) \nsubseteq T$, and $y \in R(q_v)$. Note that since $y \notin Q$, we have $y \neq q_v$. Because $q_v \notin T$, we obtain $v \neq q_v$ also. So, there exist v, q_v, y such that $v \neq q_v$, $q_v \neq y$, $v \, R \, q_v$, and $q_v \, R \, y$.

Because $v \in T = T^{\blacktriangledown\blacktriangle}$, there is $a \in T^{\blacktriangledown}$ such that $a \, R \, v$. The fact that $R(v) \nsubseteq T$ gives $v \notin T^{\blacktriangledown}$ and hence we must have $a \neq v$. Observe also that $R(y) \subseteq Q^{\blacktriangle}$ is not possible. This is because $y \notin Z^{\blacktriangle} = (S \setminus Q)^{\blacktriangle} \cup T^{\blacktriangle} \cup P^{\blacktriangle}$, that is, $y \notin (S \setminus Q)^{\blacktriangle}$, $y \notin T^{\blacktriangle}$, and $y \notin P^{\blacktriangle}$, combined with $y \in S^{\blacktriangle} \cup T^{\blacktriangle} = Y$, yield $y \in Y \setminus ((S \setminus Q)^{\blacktriangle} \cup T^{\blacktriangle})$. Hence, by (4.19), $R(y) \subseteq Q^{\blacktriangle}$ would imply $y \in P \subseteq P^{\blacktriangle}$, a contradiction. Therefore, $R(y) \nsubseteq Q^{\blacktriangle} = \bigcup_{q \in Q} R(q)$, and so there is an element $u \in R(y)$ such that

$$(\forall q \in Q) \, u \notin R(q). \tag{4.22}$$

Then $y \, R \, u$, and clearly $u \neq y$, because $y \, R \, q_v$ holds.

We need to prove there are no R-related elements in the R-path (a, v, q_v, y, u) except two consecutive ones. If this is true, then all the elements of the path are distinct, because $a \neq v$, $v \neq q_v$, $q_v, \neq y$, $y \neq u$ and R is reflexive. Since this is a contradiction to our assumption that R satisfies (C), there is no $y \in (S^{\blacktriangle} \cup T^{\blacktriangle}) \setminus Z^{\blacktriangle}$, and we may conclude that $Z^{\blacktriangle} = S^{\blacktriangle} \cup T^{\blacktriangle} = Y$, which finishes the proof.

Indeed, $a \, R \, q_v$ implies $q_v \in R(a) \subseteq T^{\blacktriangledown\blacktriangle} = T$, contradicting $q_v \notin T$. Similarly, $a \, R \, y$ and $v \, R \, y$ are not possible, because $a, v \in T$ and $y \notin T^{\blacktriangle}$. By (4.22), $q_v \, R \, u$ cannot hold. Furthermore, $a \, R \, u$ implies $u \in R(a) \subseteq T$ and $y \in R(u) \subseteq T^{\blacktriangle}$, a contradiction. Finally, since $R(v) \subseteq S \cup T$, $v \, R \, u$ implies $u \in R(v) \subseteq S \cup T$. Moreover, we get $u \in (S \setminus Q) \cup T$, because (4.22) implies $u \notin Q$. So, this yields $y \in R(u) \subseteq (S \setminus Q)^{\blacktriangle} \cup T^{\blacktriangle} \subseteq Z^{\blacktriangle}$, a contradiction again. Hence, neither $v \, R \, u$ is possible. □

Lemma 4.12. *Any tolerance R on U satisfies* (C) *if and only if for any $X \subseteq U$, ${R_X}^3$ is an equivalence on X.*

Proof. The relation ${R_X}^3$ is a tolerance on any $X \subseteq U$ and ${R_X}^3 \subseteq {R_X}^4$. If ${R_X}^3 \subset {R_X}^4$, then there is an R-path $(a_0, ..., a_4)$ of length 4 in X with $(a_0, a_4) \notin {R_X}^3$, and hence (C) fails. This means that (C) must imply ${R_X}^4 = R_X^3$. Additionally, we can see by induction that (C) implies ${R_X}^n = {R_X}^3$ for all $n \geq 3$. Then, ${R_X}^3 \circ {R_X}^3 = {R_X}^6 = {R_X}^3$ and hence the tolerance ${R_X}^3$ is transitive, that is, ${R_X}^3$ is an equivalence. Conversely, let $(a_0, ..., a_4)$ be an R-path of length 4, $X = \{a_0, ..., a_4\}$, and suppose that ${R_X}^3$ is an equivalence. Then $(a_0, a_4) \in {R_X}^4 \subseteq {R_X}^6 = {R_X}^3 \circ {R_X}^3 \subseteq {R_X}^3$ implies $(a_0, a_4) \in {R_X}^3$. Observe that this is possible only if condition (C) holds. □

Corollary 4.13. *Let R be a tolerance on U. Then, the following are equivalent:*

(a) *RS_X is a complete lattice for all $X \subseteq U$.*
(b) *RS_X is a lattice for all $X \subseteq U$ with $|X| = 5$.*
(c) *For any $X \subseteq U$, ${R_X}^3$ is an equivalence on X.*

Proof. The implication (a)⇒(b) is trivial. If (b) holds, then R satisfies condition (C) according to Lemma 4.10. Hence, by Lemma 4.12, every ${R_X}^3$ is an equivalence, and we have (b)⇒(c). Again, by Lemma 4.12, (c) implies that R_X satisfies (C) for all $X \subseteq U$. Hence, by applying Theorem 4.11 for each $X \subseteq U$ and R_X, we obtain (a), and so (c)⇒(a). □

5. Disjoint representation of rough sets

Disjoint representations of rough sets were introduced by P. Pagliani in [14]. Each rough set $(X^\blacktriangledown, X^\blacktriangle)$ may as well be represented as a pair $(X^\blacktriangledown, X^{\blacktriangle c})$, called the *disjoint rough set* of X. Clearly, $(X^\blacktriangledown, X^{\blacktriangle c}) \in \wp(U)^\blacktriangledown \times \wp(U)^\blacktriangledown$ and now $X^{\blacktriangle c}$ can be interpreted as the set of elements that certainly are outside X, while $X^\blacktriangledown$ consists of elements certainly belonging to X. Let us denote

$$dRS = \{(X^\blacktriangledown, X^{\blacktriangle c}) \mid X \subseteq U\},$$

and define an order-isomorphism ϕ between $\wp(U)^\blacktriangledown \times \wp(U)^\blacktriangle$ and $\wp(U)^\blacktriangledown \times \wp(U)^{\blacktriangledown\mathrm{op}}$ by $(A,B) \mapsto (A, B^c)$. Obviously, ϕ is also an order-isomorphism between RS and dRS, when dRS is ordered by the order of $\wp(U)^\blacktriangledown \times \wp(U)^{\blacktriangledown\mathrm{op}}$. We define a De Morgan operation $\mathfrak{c}$ on $\wp(U)^\blacktriangledown \times \wp(U)^{\blacktriangledown\mathrm{op}}$ by

$$\mathfrak{c}\colon (A,B) \to (B,A). \tag{5.1}$$

Clearly, for all $(A,B) \in \wp(U)^\blacktriangledown \times \wp(U)^\blacktriangle$,

$$\phi({\sim}(A,B)) = \phi(B^c, A^c) = (B^c, A) = \mathfrak{c}(A, B^c) = \mathfrak{c}(\phi(A,B)),$$

where $\sim$ is the De Morgan operation on $\wp(U)^\blacktriangledown \times \wp(U)^\blacktriangle$ defined in (4.3). Additionally, if $(X^\blacktriangledown, X^{\blacktriangle c}) \in dRS$, then $\mathfrak{c}(X^\blacktriangledown, X^{\blacktriangle c}) = (X^{c\blacktriangledown}, X^{c\blacktriangle c}) \in dRS$.

In [14] Pagliani showed that in the case of equivalences, disjoint rough sets are closely connected to the construction of Nelson algebras by Sendlewski [17]. Pagliani's results are generalized for quasiorders in [10], where it is proved that for any quasiorder R on U,

$$dRS = \{(A,B) \in \wp(U)^\blacktriangledown \times \wp(U)^\blacktriangledown \mid A \cap B = \emptyset \text{ and } \mathcal{S} \subseteq A \cup B\},$$

where $\mathcal{S}$ is the set of singleton $R(x)$-sets defined as in (4.4). By applying this equality it is possible to show that on dRS, and thus on RS, a Nelson algebra can be defined by applying Sendlewski's construction. However, in the case of tolerances the situation is quite different, because RS and dRS do not always form lattices, and even they do, the lattices are not necessarily distributive. However, in case the tolerance R induced by an irredundant covering of U, these lattices are distributive, and a quasi-Nelson algebra can be defined on RS and dRS, as shown in Proposition 4.9. Anyway, these quasi-Nelson algebras are not necessarily Nelson algebras.

In Section 4, we defined the increasing representation of rough sets, that is,

$$\mathcal{I}(RS) = \{(A,B) \in \wp(U)^\blacktriangledown \times \wp(U)^\blacktriangle \mid A^\blacktriangle \subseteq B^\blacktriangledown \text{ and } \mathcal{S} \subseteq A \cup B^c\},$$

and showed that this is the Dedekind–MacNeille completion of RS. If we map the set $\mathcal{I}(RS)$ by the isomorphism ϕ, we obtain the set

$$\mathcal{D}(RS) = \{(A,B) \in \wp(U)^\blacktriangledown \times \wp(U)^\blacktriangledown \mid A^\blacktriangle \cap B^\blacktriangle = \emptyset \text{ and } \mathcal{S} \subseteq A \cup B\}.$$

The set $\mathcal{D}(RS)$ is called the *disjoint representation of rough sets*. Obviously, the map $\mathfrak{c}$ defined in (5.1) is a De Morgan operation on $\mathcal{D}(RS)$, and if RS is a complete lattice, then RS and $\mathcal{D}(RS)$ can be identified by the map ϕ.

We end this work by studying the connection between $\mathcal{D}(RS)$ and the concept lattice $\mathfrak{B}(\mathbb{K})$ defined by the context $\mathbb{K} = (U, U, R^c)$. In [13, 20], it is considered for a concept (A, B) of an arbitrary context its *weak negation* by

$$(A, B)^{\triangle} = (A^{c\prime\prime}, A^{c\prime})$$

and its *weak opposition* by

$$(A, B)^{\triangledown} = (B^{c\prime}, B^{c\prime\prime}).$$

Especially, we are here considering the weak opposition operation $^{\triangledown}$, which satisfies for all concepts (A, B) and (C, D):

$$(A, B) \leq (C, D)^{\triangledown} \iff (C, D) \leq (A, B)^{\triangledown}.$$

We already noted in Section 3 that the concept lattice of the context $\mathbb{K} = (U, U, R^c)$ is $\mathfrak{B}(\mathbb{K}) = \{(A, A^{\top}) \mid A \in \wp(U)^{\blacktriangledown}\}$, where $^{\top}$ is the orthocomplement operation of $\wp(U)^{\blacktriangledown}$. Recall that $A^{\top} = A' = A^{\blacktriangle c} = A^{c\blacktriangledown}$. For $(A, B) \in \mathfrak{B}(\mathbb{K})$, the weak negation and the weak opposition are then defined by

$$(A, B)^{\triangle} = (A^{\blacktriangledown\top}, A^{\blacktriangledown}) \ \text{ and } \ (A, B)^{\triangledown} = (B^{\blacktriangledown}, B^{\blacktriangledown\top}).$$

We now consider the complete lattice $\mathfrak{B}(\mathbb{K}) \times \mathfrak{B}(\mathbb{K})^{\mathrm{op}}$, where $\mathfrak{B}(\mathbb{K})^{\mathrm{op}}$ is the dual of the concept lattice $\mathfrak{B}(\mathbb{K})$, that is, $\mathfrak{B}(\mathbb{K})^{\mathrm{op}}$ is ordered by

$$(A_1, B_1) \leq (A_2, B_2) \iff A_1 \supseteq A_2 \iff B_1 \subseteq B_2.$$

Let $\mathrm{ext}(\alpha)$ denote the extent A of a concept $\alpha = (A, B)$. We define the set

$$\mathcal{FC}(RS) = \{(\alpha, \beta) \in \mathfrak{B}(\mathbb{K}) \times \mathfrak{B}(\mathbb{K}) \mid \beta \leq \alpha^{\triangledown} \text{ in } \mathfrak{B}(\mathbb{K}) \text{ and } \mathcal{S} \subseteq \mathrm{ext}(\alpha) \cup \mathrm{ext}(\beta)\},$$

and we call it the *formal concept representation of rough sets.* We order the set $\mathcal{FC}(RS)$ by the order of $\mathfrak{B}(\mathbb{K}) \times \mathfrak{B}(\mathbb{K})^{\mathrm{op}}$.

Proposition 5.1. *Let R be a tolerance on U and $\mathbb{K} = (U, U, R^c)$.*
(a) *$\mathcal{FC}(RS)$ is a complete sublattice of $\mathfrak{B}(\mathbb{K}) \times \mathfrak{B}(\mathbb{K})^{\mathrm{op}}$.*
(b) *The complete lattices $\mathcal{FC}(RS)$ and $\mathcal{D}(RS)$ are isomorphic.*

Proof. First, let us define a map $\varphi \colon \wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangledown} \to \mathfrak{B}(\mathbb{K}) \times \mathfrak{B}(\mathbb{K})$ by setting

$$(A, B) \mapsto ((A, A^{\top}), (B, B^{\top})).$$

Trivially, the map φ is well defined. Next we show that φ is an order-isomorphism between the complete lattices $\wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangledown\mathrm{op}}$ and $\mathfrak{B}(\mathbb{K}) \times \mathfrak{B}(\mathbb{K})^{\mathrm{op}}$.

If $(A, B), (C, D) \in \wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangledown}$, then

$$\begin{aligned}
&(A, B) \leq (C, D) \text{ in } \wp(U)^{\blacktriangledown} \times \wp(U)^{\blacktriangledown\mathrm{op}} && \iff \\
&A \subseteq C \text{ and } B \supseteq D && \iff \\
&(A, A^{\top}) \leq (C, C^{\top}) \text{ and } (B, B^{\top}) \geq (D, D^{\top}) \text{ in } \mathfrak{B}(\mathbb{K}) && \iff \\
&\varphi(A, B) \leq \varphi(C, D) \text{ in } \mathfrak{B}(\mathbb{K}) \times \mathfrak{B}(\mathbb{K})^{\mathrm{op}}
\end{aligned}$$

Thus, φ is an order-embedding. If $((A, A^{\top}), (B, B^{\top})) \in \mathfrak{B}(\mathbb{K}) \times \mathfrak{B}(\mathbb{K})$, then $A, B \in \wp(U)^{\blacktriangledown}$ and $\varphi(A, B) = ((A, A^{\top}), (B, B^{\top}))$. Therefore, the map φ is also onto, and it is an order-isomorphism.

Next, we prove that $\mathcal{FC}(RS)$ is the image of $\mathcal{D}(RS)$ under φ. Note first that for all $A, B \in \wp(U)^{\blacktriangledown}$,

$$A^{\blacktriangle} \cap B^{\blacktriangle} = \emptyset \iff A^{\blacktriangle\blacktriangle} \cap B = \emptyset \iff B \subseteq A^{\blacktriangle\blacktriangle c} = A^{c\blacktriangledown\blacktriangledown} = A^{\top\blacktriangledown}.$$

Since $(A, A^\top)^\triangledown = (A^{\top\blacktriangledown}, A^{\top\blacktriangledown\top})$, we have that

$$A^\blacktriangle \cap B^\blacktriangle = \emptyset \iff (B, B^\top) \leq (A, A^\top)^\triangledown \text{ in } \mathfrak{B}(\mathbb{K}).$$

Additionally, $\mathcal{S} \subseteq A \cup B = \mathrm{ext}(A, A^\top) \cup \mathrm{ext}(B, B^\top)$. These facts imply

$$(A, B) \in \mathcal{D}(RS) \iff \varphi(A, B) \in \mathcal{FC}(RS).$$

Since φ is a bijection, we get

$$\mathcal{FC}(RS) = (\varphi \circ \varphi^{-1})(\mathcal{FC}(RS)) = \varphi(\mathcal{D}(RS)).$$

Hence, φ determines an order-isomorphism between the complete lattices $\mathcal{D}(RS)$ and $\mathcal{FC}(RS)$, which proves (b).

Because $\mathcal{I}(RS)$ is a complete sublattice of $\wp(U)^\blacktriangledown \times \wp(U)^\blacktriangle$ by Proposition 4.4(b), its image $\mathcal{D}(RS)$ under the isomorphism $\phi\colon (A, B) \mapsto (A, B^c)$ is a complete sublattice of $\wp(U)^\blacktriangledown \times \wp(U)^{\blacktriangledown\mathrm{op}}$. This implies that the image $\mathcal{FC}(RS)$ of $\mathcal{D}(RS)$ under φ is a complete sublattice of $\mathfrak{B}(\mathbb{K}) \times \mathfrak{B}(\mathbb{K})^{\mathrm{op}}$, and claim (a) is proved. □

Since for all $(\alpha, \beta) \in \mathcal{FC}(RS)$, $\beta \leq \alpha^\triangledown \iff \alpha \leq \beta^\triangledown$, it is easy to see that $\mathfrak{c}^*\colon (\alpha, \beta) \mapsto (\beta, \alpha)$ is a De Morgan operation on $\mathcal{FC}(RS)$. Up to isomorphism, this operation is the same as in $\mathcal{I}(RS)$ and $\mathcal{D}(RS)$. Namely, if $(A, B) \in \mathcal{D}(RS)$, then

$$\varphi(\mathfrak{c}(A, B)) = \varphi(B, A) = ((B, B^\top), (A, A^\top)) = \mathfrak{c}^*((A, A^\top), (B, B^\top)) = \mathfrak{c}^*(\varphi(A, B)).$$

We conclude this work by giving the following summary of rough representations:

- For any tolerance R on U, the representations $\mathcal{I}(RS)$, $\mathcal{D}(RS)$, and $\mathcal{FC}(RS)$ are Dedekind–MacNeille completions of RS equipped with De Morgan operations satisfying (K3) that are identical up to isomorphism.
- The ordered sets RS and dRS are isomorphic, and they are complete lattices if and only if RS is a complete sublattice of $\wp(U)^\blacktriangledown \times \wp(U)^\blacktriangle$, or, equivalently, dRS is a complete sublattice of $\wp(U)^\blacktriangledown \times \wp(U)^{\blacktriangledown\mathrm{op}}$.
- If RS and dRS are complete lattices, then they are identical to $\mathcal{I}(RS)$ and $\mathcal{D}(RS)$, respectively. This implies also $RS \cong \mathcal{FC}(RS)$.
- If R induced by an irredundant covering of U, then RS, dRS, $\mathcal{I}(RS)$, $\mathcal{D}(RS)$, and $\mathcal{FC}(RS)$ determine isomorphic quasi-Nelson algebras.

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